Blow-up solutions for two coupled Gross-Pitaevskii equations with attractive interactions*

Yujin Guo[a] , Xiaoyu Zeng[b] and Huan-Song Zhou[b†]
[a]Wuhan Institute of Physics and Mathematics, Chinese Academy of Sciences, P.O. Box 71010, Wuhan 430071, P. R. China
[b]Department of Mathematics, Wuhan University of Technology, Wuhan 430070, China

Abstract

The paper is concerned with a system of two coupled time-independent Gross-Pitaevskii equations in $\mathbb{R}^2$, which is used to model two-component Bose-Einstein condensates with both attractive intraspecies and attractive interspecies interactions. This system is essentially an eigenvalue problem of a stationary nonlinear Schrödinger system in $\mathbb{R}^2$, solutions of the problem are obtained by seeking minimizers of the associated variational functional with constrained mass (i.e. L^2-norm constaints). Under certain type of trapping potentials $V_i(x)$ $(i = 1, 2)$, the existence, non-existence and uniqueness of this kind of solutions are studied. Moreover, by establishing some delicate energy estimates, we show that each component of the solutions blows up at the same point (i.e., one of the global minima of $V_i(x)$) when the total interaction strength of intraspecies and interspecies goes to a critical value. An optimal blowing up rate for the solutions of the system is also given.

Keywords: Schrödinger equations; Gross-Pitaevskii equation; elliptic systems; constrained minimization; blow up.

*MSC:*35J50,35J61,35J47,47J30,35Q40.

1 Introduction

In this paper, we study the following system of two coupled time-independent Gross-Pitaevskii equations:

$$\begin{cases} -\Delta u_1 + V_1(x)u_1 = \mu_1 u_1 + b_1 u_1^3 + \beta u_2^2 u_1 & \text{in } \mathbb{R}^2, \\ -\Delta u_2 + V_2(x)u_2 = \mu_2 u_2 + b_2 u_2^3 + \beta u_1^2 u_2 & \text{in } \mathbb{R}^2. \end{cases} \tag{1.1}$$

The system (1.1) arises in describing two-component Bose-Einstein condensates (BEC), where $(V_1(x), V_2(x))$ is a certain type of trapping potentials, $(\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R}$ is the

*Email: yjguo@wipm.ac.cn; xyzeng@whut.edu.cn; hszhou@whut.edu.cn
†Corresponding author.

chemical potential, b_i ($i = 1, 2$) and β are the interaction strength of cold atoms inside each component (i.e. intraspecies) and between two components (i.e. interspecies), respectively. Here $b_i > 0$ (or $\beta > 0$) represents the intraspecies (or interspecies) interaction is attractive, otherwise, it is repulsive. Much attention has been paid to the experimental studies of BECs since the BEC phenomena were successfully observed in 1995 in the pioneering experiments [1, 12]. After that various BEC phenomena are observed in BEC experiments such as the symmetry breaking, the collapse, the appearance of quantized vortices in rotating traps, the phase segregation, etc., which inspired the theoretical investigation on BECs, specially on the Gross-Pitaevskii equations–the fundamental model of describing the BEC. Multiple-component BECs can display more interesting phenomena absent in single-component BEC. In the past decade, under variant conditions on $V_i(x)$, b_i and β, the analogues of (1.1) have been investigated widely, see e.g. [3, 5, 9, 10, 11, 7, 18, 19, 24, 25, 29, 31, 33, 36, 40] and the references therein. In these mentioned papers, the authors are concerned with either the semiclassical states of the system (i.e. replacing $-\Delta$ by $-\varepsilon\Delta$ in (1.1)) [25, 29, 19, 10, 31], or the problem (1.1) with $V_i(x)$ being constants [5, 9, 11, 27, 38], or the problem (1.1) in repulsive cases [3, 7, 18, 24, 33], etc. The results in these papers are mainly on studying the existence of positive solutions for small $\varepsilon > 0$, the location of the maximum point of the solutions and the behavior of the solutions with respect to $\varepsilon \to 0$ or the coupling parameter tends to infinity. To the authors' knowledge, there seems few results concerning the blowing-up analysis on the L^2- normalized solutions of (1.1) with $b_i > 0$.

In this paper, we are interested in dealing with the blowing-up properties of solutions for system (1.1) with both attractive intraspecies and interspecies interactions, that is, $b_i > 0$ and $\beta > 0$ (totally attractive case). We leave the case of the attractive intraspecies (i.e. $b_i > 0$) and the repulsive interspecies (i.e. $\beta < 0$) to the companion work [17]. The case of repulsive intraspecies and interspecies interactions was studied recently in [24, 33] and the references therein. When $b_i > 0$ and $\beta > 0$, as mentioned in [33], (1.1) is certainly a very different problem because the energy functional may not be bounded from below. In totally attractive case, we may expect from the single component BEC, see e.g., [12, 14, 16], that the collapse still happens if the particle number increases beyond a critical value. In addition to the intraspecies interaction among atoms in each component, there exist interspecies interactions among the components for multiple-component BECs. Therefore, multiple-component BECs present more complicated characters than single component BEC, and the corresponding analytic investigations are more challenging.

It is well-known that the system (1.1) can be obtained from the associated time dependent nonlinear Schrödinger equations if one seeks for the following type standing-wave solutions

$$(\psi_1(x,t), \psi_2(x,t)) = (u_1(x)e^{-\mathbf{i}\mu_1 t}, u_2(x)e^{-\mathbf{i}\mu_2 t}), \text{ where } \mathbf{i}^2 = -1.$$

(1.1) is essentially an eigenvalue problem of a system of two stationary nonlinear Schrödinger equations, which is also the system of Euler-Lagrange equations (μ_1, μ_2 are the Lagrange multipliers) of the following constrained minimization problem,

$$\hat{e}(b_1, b_2, \beta) := \inf_{\{(u_1,u_2)\in\mathcal{M}\}} E_{b_1,b_2,\beta}(u_1, u_2), \tag{1.2}$$

where $\mathcal{M}$ is the so-called mass constraint

$$\mathcal{M} = \Big\{(u_1, u_2) \in \mathcal{X} : \int_{\mathbb{R}^2} |u_1|^2 dx = \int_{\mathbb{R}^2} |u_2|^2 dx = 1\Big\} \tag{1.3}$$

and $\mathcal{X} = \mathcal{H}_1 \times \mathcal{H}_2$ with

$$\begin{aligned} \mathcal{H}_i &= \Big\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V_i(x)|u(x)|^2 dx < \infty\Big\}, \\ \|u\|_{\mathcal{H}_i} &= \Big(\int_{\mathbb{R}^2} \Big[|\nabla u|^2 + V_i(x)|u(x)|^2\Big] dx\Big)^{\frac{1}{2}}, \text{ where } i = 1, 2. \end{aligned} \tag{1.4}$$

The energy functional $E_{b_1,b_2,\beta}(u_1, u_2)$ is given by

$$\begin{aligned} E_{b_1,b_2,\beta}(u_1, u_2) =& \sum_{i=1}^{2} \int_{\mathbb{R}^2} \Big(|\nabla u_i|^2 + V_i(x)|u_i|^2 - \frac{b_i}{2}|u_i|^4\Big) dx \\ & - \beta \int_{\mathbb{R}^2} |u_1|^2|u_2|^2 dx\,, \quad (u_1, u_2) \in \mathcal{X}. \end{aligned} \tag{1.5}$$

In this paper, we only interested in the solutions of (1.1) with constrained mass, that is, the minimizers of (1.2). We mention that, different from the single component minimization problem (see, e.g. [15]), in the two-component case it seems not clear whether a minimizer of (1.2) is a ground state of (1.1). But we know that if (u_1, u_2) is a minimizer of (1.2), then (u_1, u_2) is a positive solution of (1.1) for some Lagrange multiplier $(\mu_1, \mu_2) \in \mathbb{R} \times \mathbb{R}$. In this paper, $V_i(x)$ are trapping potentials of the following type

$$V_i(x) \in L^\infty_{\text{loc}}(\mathbb{R}^2), \ \lim_{|x|\to\infty} V_i(x) = \infty \ \text{ and } \ \inf_{x\in\mathbb{R}^2} V_i(x) = 0, \ \ i = 1, 2. \tag{1.6}$$

Throughout the paper, we assume that both $\inf_{x\in\mathbb{R}^2} \big(V_1(x)+V_2(x)\big)$ and $\inf_{x\in\mathbb{R}^2} V_i(x)$ are attained. Since problem (1.2) is invariant by adding suitable constants to $V_i(x)$ and we may simply assume that $\inf_{x\in\mathbb{R}^2} V_i(x) = 0$. Moreover, in what follows we assume that $b_i > 0$ ($i = 1, 2$) and $\beta > 0$, and denote the norm of $L^p(\mathbb{R}^2)$ by $\|\cdot\|_p$ for $p \in (1, \infty)$.

We study problem (1.2) which is motivated by the recent works [4, 14, 15, 16], etc., where the following single component minimization problem was investigated:

$$e_i(a) := \inf_{\{u\in\mathcal{H}_i, \|u\|_2^2=1\}} E^i_a(u), \ \ a > 0, \tag{1.7}$$

and the energy functional $E^i_a(u)$ satisfies

$$E^i_a(u) := \int_{\mathbb{R}^2} \big(|\nabla u(x)|^2 + V_i(x)|u(x)|^2\big) dx - \frac{a}{2} \int_{\mathbb{R}^2} |u(x)|^4 dx, \ \ i = 1 \text{ or } 2. \tag{1.8}$$

Actually, it was proved in [4, 14] that (1.7) admits minimizers if and only if $0 < a < a^* := \|Q\|_2^2$, where $Q = Q(|x|) > 0$ denotes the unique positive solution of the following nonlinear scalar field equation

$$-\Delta u + u - u^3 = 0 \text{ in } \mathbb{R}^2, \ \text{ where } \ u \in H^1(\mathbb{R}^2). \tag{1.9}$$

On the other hand, the following Gagliardo-Nirenberg inequality was shown in [39] that

$$\int_{\mathbb{R}^2} |u(x)|^4 dx \le \frac{2}{\|Q\|_2^2} \int_{\mathbb{R}^2} |\nabla u(x)|^2 dx \int_{\mathbb{R}^2} |u(x)|^2 dx, \quad u \in H^1(\mathbb{R}^2), \tag{1.10}$$

where the identity is achieved at $u(x) = Q(|x|)$. One can note from (1.9) that $Q(|x|)$ satisfies

$$\int_{\mathbb{R}^2} |\nabla Q|^2 dx = \int_{\mathbb{R}^2} Q^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} Q^4 dx, \tag{1.11}$$

see also Lemma 8.1.2 in [8]. Moreover, we have

$$Q(x)\,,\ |\nabla Q(x)| = O(|x|^{-\frac{1}{2}} e^{-|x|}) \text{ as } |x| \to \infty, \tag{1.12}$$

see [13, Proposition 4.1] for more details.

We now introduce briefly the main results of the present paper. Our first result is concerned with the following existence and nonexistence of minimizers for (1.2).

Theorem 1.1. *Let Q be the unique positive radial solution of (1.9) and $a^* = \|Q\|_2^2$. If $V_i(x)$ satisfies (1.6) for $i = 1, 2$. Then,*

(i). *When $0 < b_1 < a^*$, $0 < b_2 < a^*$ and $\beta < \sqrt{(a^* - b_1)(a^* - b_2)}$, problem (1.2) has at least one minimizer.*

(ii). *Either $b_1 > a^*$ or $b_2 > a^*$ or $\beta > \frac{a^*-b_1}{2} + \frac{a^*-b_2}{2}$, (1.2) has no minimizer.*

In order to prove Theorem 1.1, in Section 2 we introduce the following minimization problem

$$\mathcal{O}(b_1, b_2, \beta) := \inf_{\left\{ \substack{u_i \,\in\, H^1(\mathbb{R}^2), \\ \|u_i\|_2^2 \,=\, 1,} \quad i=1,2 \right\}} \frac{\int_{\mathbb{R}^2} (|\nabla u_1|^2 + |\nabla u_2|^2) dx}{\frac{b_1}{2} \int_{\mathbb{R}^2} |u_1|^4 dx + \frac{b_2}{2} \int_{\mathbb{R}^2} |u_2|^4 dx + \beta \int_{\mathbb{R}^2} |u_1|^2 |u_2|^2 dx}. \tag{1.13}$$

We shall prove in Proposition 2.1 that if $\mathcal{O}(b_1, b_2, \beta) > 1$, then there exists at least one minimizer for (1.2). However, if $\mathcal{O}(b_1, b_2, \beta) < 1$, then there is no minimizer for (1.2). Therefore, by Proposition 2.1, to establish Theorem 1.1 it suffices to evaluate $\mathcal{O}(b_1, b_2, \beta)$, depending on the range of (b_1, b_2, β). On the other hand, as shown in Lemma A.2 of the appendix, Theorem 1.1 can be proved alternatively by applying directly the Gagliardo-Nirenberg inequality (1.10) and some recaling techniques. We remark that some results similar to Theorem 1.1 were also proved in [3, Theorem 2.6] by this kind of ideas.

Theorem 1.1 gives a complete classification of the existence and nonexistence of minimizers for (1.2), except that (b_1, b_2, β) satisfies

$$0 < b_1 \le a^*, \ \ 0 < b_2 \le a^* \text{ and } \beta \in \big[\sqrt{(a^* - b_1)(a^* - b_2)}, \frac{a^* - b_1}{2} + \frac{a^* - b_2}{2}\big]. \tag{1.14}$$

When (b_1, b_2, β) satisfies (1.14), in general it is difficult to evaluate $\mathcal{O}(b_1, b_2, \beta)$ so that one cannot employ directly Proposition 2.1 to discuss the existence of minimizers for (1.2). Therefore, some new ideas are needed to address this case. Under some additional assumptions on (b_1, b_2, β), in this paper we shall derive the following existence and nonexistence of minimizers for (1.2).

Theorem 1.2. *Under condition (1.6) and let* $\inf_{x\in\mathbb{R}^2}\big(V_1(x)+V_2(x)\big)$ *be attained. If* $0 < b_1 \neq b_2 < a^*$ *satisfies additionally*

$$|b_1 - b_2| \leq 2\sqrt{(a^*-b_1)(a^*-b_2)}, \tag{1.15}$$

then there exists $\delta = \delta(b_1, b_2) \in (\sqrt{(a^*-b_1)(a^*-b_2)}, \frac{2a^*-b_1-b_2}{2}]$ *such that*

$$\text{for any } \beta \in [\sqrt{(a^*-b_1)(a^*-b_2)}, \delta(b_1,b_2)\big),$$

there exists at least one minimizer for (1.2).

Theorem 1.3. *Suppose* $V_i(x)$ $(i = 1, 2)$ *satisfies (1.6) and assume* $\inf_{x\in\mathbb{R}^2}\big(V_1(x)+V_2(x)\big)$ *is attained. If* $0 < b_1 = b_2 < a^*$ *and* $\beta > 0$ *satisfy (1.14). Then*

(i) *Problem (1.2) has no minimizer if* $\inf_{x\in\mathbb{R}^2}\big(V_1(x)+V_2(x)\big) = 0$.

(ii) *Problem (1.2) has at least one minimizer if*

$$\hat{e}(a^*-\beta, a^*-\beta, \beta) < \inf_{x\in\mathbb{R}^2}\big(V_1(x)+V_2(x)\big).$$

We mention that, for $b_1 = b_2 \in (0, a^*)$, condition (1.14) implies $\beta = a^* - b_1 = a^* - b_2$, then the point (b_1, b_2, β) must be located on the segment $(a^*-\beta, a^*-\beta, \beta)$ with $\beta \in (0, a^*)$.

In view of Theorems 1.1 to 1.3, the existence and nonexistence of minimizers for (1.2) still remain open for the case where $0 < b_1 \neq b_2 \leq a^*$ and $\beta > 0$ is close to $\frac{2a^*-b_1-b_2}{2}$ from below. By applying Proposition 2.1, we shall derive Theorem 1.2 through establishing the estimate $\mathcal{O}(b_1, b_2, \beta) > 1$ under the additional assumption (1.15). As for the case where $0 < b_1 = b_2 < a^*$, it follows from (2.2) that $\mathcal{O}(a^*-\beta, a^*-\beta, \beta) = 1$, and we shall give the proof of Theorem 1.3 (ii) by applying Ekeland's variational principle [35, Theorem 5.1].

The proof of Lemma A.2 in the appendix implies that the following relationship always holds

$$0 \leq \hat{e}(a^*-\beta, a^*-\beta, \beta) \leq \inf_{x\in\mathbb{R}^2}\big(V_1(x)+V_2(x)\big). \tag{1.16}$$

In particular, the second inequality in (1.16) can be strict for certain potentials $V_1(x)$ and $V_2(x)$. Here is an example:

Example 1.1. For any given points x_1 and x_2 in $\mathbb{R}^2$ satisfying $|x_1 - x_2| > 4$, consider the function $0 \leq \zeta_i(x) \in C_0^2(B_1(x_i))$ satisfying $\|\zeta_i\|_2 = 1$, where $i = 1, 2$, and define a positive constant C_ζ by

$$C_\zeta := \sum_{i=1}^{2}\int_{\mathbb{R}^2}\Big(|\nabla\zeta_i|^2 - \frac{a^*-\beta}{2}|\zeta_i|^4\Big)dx - \beta\int_{\mathbb{R}^2}|\zeta_1|^2|\zeta_2|^2dx < \infty.$$

Let $0 \leq V_i(x) \in C^2(\mathbb{R}^2;\mathbb{R})$ satisfy $V_i(x) = 0$ in $B_1(x_i)$ and $V_i(x) \geq 2C_\zeta$ in $B_2^c(x_i)$ as well as $\lim_{|x|\to\infty} V_i(x) = \infty$, where $i = 1, 2$. One can check that

$$\hat{e}(a^*-\beta, a^*-\beta, \beta) \leq E_{a^*-\beta, a^*-\beta, \beta}(\zeta_1, \zeta_2) = C_\zeta < 2C_\zeta \leq \inf_{x\in\mathbb{R}^2}\big(V_1(x)+V_2(x)\big).$$

Example 1.1 and Theorem 1.3 (ii) show us that for any fixed $0 < \beta < a^*$, there exists at least one minimizer of (1.2) at $(b_1, b_2) = (a^* - \beta, a^* - \beta)$ for some suitable potentials $V_1(x)$ and $V_2(x)$. On the other hand, Theorem 1.3 (i) gives the non-existence of minimizers for (1.2) at $(b_1, b_2) = (a^* - \beta, a^* - \beta)$ and $0 < \beta < a^*$ for the case where $\inf_{x\in\mathbb{R}^2}\big(V_1(x) + V_2(x)\big) = 0$ is attained.

Without loss of generality, in the follows one may restrict the minimization of (1.2) to non-negative vector functions (u_1, u_2), since $E_{b_1,b_2,\beta}(u_1, u_2) \geq E_{b_1,b_2,\beta}(|u_1|, |u_2|)$ for any $(u_1, u_2) \in \mathcal{X}$, due to the fact that $\nabla|u_i| \leq |\nabla u_i|$ a.e. in $\mathbb{R}^2$ ($i = 1, 2$). We next discuss the uniqueness of non-negative minimizers for (1.2). Our recent results in [14, 16] show that the single component minimization problem (1.7) has a unique non-negative minimizer when the parameter $a > 0$ is suitably small, and however there may exist multiple non-negative minimizers for (1.7) as $a \nearrow a^*$ under some classes of trapping potentials. We shall prove in Section 4 that a similar uniqueness result also holds for (1.2) if $|(b_1, b_2, \beta)|$ is suitably small.

Theorem 1.4. *If $V_i(x)$ satisfies (1.6) for $i = 1$ and 2, then (1.2) admits a unique non-negative minimizer if $|(b_1, b_2, \beta)|$ is suitably small.*

A similar result on uniqueness for the single component problem (1.7) was proved in [2, 28] by using the contracting map. However, this method seems not work for our problem. In this paper, we prove Theorem 1.4 by employing an implicit function theorem.

For any fixed $0 < \beta < a^*$, we finally focus on the limit behavior of the minimizers for (1.2) as $(b_1, b_2) \nearrow (a^* - \beta, a^* - \beta)$, i.e., $(b_1 + \beta, b_2 + \beta) \nearrow (a^*, a^*)$ in the case that there is no minimizer for (1.2) at the threshold $(b_1, b_2) = (a^* - \beta, a^* - \beta)$. In view of Theorem 1.3 (i), we shall consider the special case where $\inf\limits_{x\in\mathbb{R}^2}\big(V_1(x) + V_2(x)\big) = 0$, *i.e.*, the minima of $V_1(x)$ coincide with those of $V_2(x)$. More precisely, we assume that for $V_i(x)$ ($i = 1, 2$) takes the form of

$$\begin{aligned} &V_i(x) = h_i(x) \prod_{j=1}^{n_i} |x - x_{ij}|^{p_{ij}} \ \text{ with } \ C < h_i(x) < 1/C \ \text{ in } \mathbb{R}^2, \\ &\text{and } \ h_i(x) \in C^{\alpha}_{\text{loc}}(\mathbb{R}^2) \ \text{ for some } \ \alpha \in (0, 1), \end{aligned} \tag{1.17}$$

where $n_i \in \mathbb{N}$, $p_{ij} > 0$, $x_{ik} \neq x_{ij}$ for $k \neq j$, and $\lim\limits_{x\to x_{ij}} h_i(x)$ exists for all $1 \leq j \leq n_i$. Without loss of generality, we also assume that there exists $1 \leq l \leq \min\{n_1, n_2\}$ such that

$$\begin{aligned} &x_{1j} = x_{2j}, \quad \text{where} \quad j = 1, \ldots, l; \\ &x_{1j_1} \neq x_{2j_2}, \quad \text{where} \quad j_i \in \big\{l + 1, \ldots n_i\big\} \ \text{ and } \ i = 1,\ 2. \end{aligned} \tag{1.18}$$

Note that (1.18) implies

$$\Lambda := \big\{x \in \mathbb{R}^2 : V_1(x) = V_2(x) = 0\big\} = \big\{x_{11}, x_{12}, \cdots, x_{1l}\big\}. \tag{1.19}$$

Define

$$\bar{p}_j := \min\big\{p_{1j}, p_{2j}\big\}, \ \ j = 1, \ldots, l; \quad p_0 := \max_{1\leq j\leq l} \min\big\{p_{1j}, p_{2j}\big\} = \max_{1\leq j\leq l} \bar{p}_j, \tag{1.20}$$

so that

$$\bar{\Lambda} := \{x_{1j} : \bar{p}_j = p_0, j = 1, \dots, l\} \subset \Lambda. \tag{1.21}$$

Let $\gamma_j \in (0, \infty]$ be given by

$$\gamma_j := \lim_{x \to x_{1j}} \frac{V_1(x) + V_2(x)}{|x - x_{1j}|^{p_0}}, \quad 1 \le j \le l. \tag{1.22}$$

Note that $\gamma_j < \infty$ if and only if $x_{1j} \in \bar{\Lambda}$, where $1 \le j \le l$. Finally, define $\gamma = \min\{\gamma_1, \dots, \gamma_l\}$ so that the set

$$\mathcal{Z} := \{x_{1j} : \ \gamma_j = \gamma, \ 1 \le j \le l\} \subset \bar{\Lambda} \tag{1.23}$$

denotes the locations of the flattest global minima of $V_1(x) + V_2(x)$. Using above notations, our main results can be stated as follows.

Theorem 1.5. *Assume that $0 < \beta < a^*$ and $V_i(x)$ satisfies (1.17)-(1.18) for $i = 1, 2$. Let (u_{b_1}, u_{b_2}) be a non-negative minimizer of (1.2) as $(b_1, b_2) \nearrow (a^* - \beta, a^* - \beta)$. Then, for any sequence $\{(b_{1k}, b_{2k})\}$ satisfying $(b_{1k}, b_{2k}) \nearrow (a^* - \beta, a^* - \beta)$ as $k \to \infty$, there exists a subsequence of $\{(b_{1k}, b_{2k})\}$, still denoted by $\{(b_{1k}, b_{2k})\}$, such that, for $i = 1$, 2, each $u_{b_{ik}}$ has a unique global maximum point $x_{ik} \overset{k}{\to} \bar{x}_0$ for some $\bar{x}_0 \in \mathcal{Z}$ and*

$$\lim_{k \to \infty} \frac{|x_{ik} - \bar{x}_0|}{\left(a^* - \frac{b_{1k} + b_{2k} + 2\beta}{2}\right)^{\frac{1}{p_0+2}}} = 0. \tag{1.24}$$

Moreover, for $i = 1$, 2,

$$\lim_{k \to \infty} \left(a^* - \frac{b_{1k} + b_{2k} + 2\beta}{2}\right)^{\frac{1}{p_0+2}} u_{b_{ik}}\left(\left(a^* - \frac{b_{1k} + b_{2k} + 2\beta}{2}\right)^{\frac{1}{p_0+2}} x + x_{ik}\right) = \frac{\lambda}{\|Q\|_2} Q(\lambda x)$$

strongly in $H^1(\mathbb{R}^2)$, where $\lambda > 0$ is given by

$$\lambda = \left(\frac{p_0 \gamma}{4} \int_{\mathbb{R}^2} |x|^{p_0} Q^2(x) dx\right)^{\frac{1}{p_0+2}} \tag{1.25}$$

for $p_0 > 0$ and $\gamma > 0$ defined in (1.20) and (1.23), respectively.

Theorems 1.4 and 1.5 imply that the symmetry breaking occurs in the minimizers of (1.2). Actually, consider the trapping potentials V_1 and V_2 of the form

$$V_1(x) = V_2(x) = \prod_{j=1}^{l} |x - x_j|^p, \quad p > 0,$$

where the points x_j with $j = 1, \cdots, l$ are arranged on the vortices of a regular polygon. Then there exist $0 < a_* \le a_{**} < a^*$ such that for $0 < b_i + \beta < a_*$ ($i = 1$ and 2), the functional (1.2) has a unique non-negative minimizer by Theorem 1.4, which has the same symmetry as that of $V_1(x) = V_2(x)$. However, when $a_{**} < b_i + \beta < a^*$ ($i = 1$, 2), we obtain from Theorem 1.5 that (1.2) possesses (at least) l different non-negative minimizers, and both components of the minimizers concentrate at a zero point

of $V_1(x) = V_2(x)$, which imply the symmetry breaking. We note that the symmetry breaking bifurcation of ground states for single nonlinear Schrödinger/Gross-Pitaevskii equations has been studied in detail in the literature, see, e.g., [20, 22, 23].

This paper is organized as follows: in Section 2 we first derive the crucial Proposition 2.1 on the auxiliary minimization problem $\mathcal{O}(b_1, b_2, \beta)$, based on which we then complete the proof of Theorem 1.1. In Section 3 we focus on the proof of Theorems 1.2 and 1.3. More exactly, we first use Proposition 2.1 to prove Theorem 1.2, and Theorem 1.3 is then proved by applying Ekeland's variational principle. Theorem 1.4 is then proved in Section 4 to address the uniqueness of nonnegative minimizers for (1.2) as $|(b_1, b_2, \beta)|$ is suitably small. In Section 5 we shall establish Proposition 5.1 on optimal energy estimates of minimizers, upon which we finally complete in Section 6 the proof of Theorem 1.5. In the appendix, we give an alternative proof of Theorem 1.1 and prove a lemma as well which is used in Section 2.

2 Existence of Minimizers

In this section, we address the proof of Theorem 1.1 on the existence of minimizers. We start with introducing the following auxiliary minimization problem

$$\mathcal{O}(b_1, b_2, \beta) := \inf_{\left\{ \substack{u_i \in H^1(\mathbb{R}^2), \\ \|u_i\|_2^2 = 1,} \quad i=1,2 \right\}} \frac{\int_{\mathbb{R}^2} (|\nabla u_1|^2 + |\nabla u_2|^2) dx}{\frac{b_1}{2} \int_{\mathbb{R}^2} |u_1|^4 dx + \frac{b_2}{2} \int_{\mathbb{R}^2} |u_2|^4 dx + \beta \int_{\mathbb{R}^2} |u_1|^2 |u_2|^2 dx}. \tag{2.1}$$

By analyzing (2.1), our first aim is to derive the following proposition, which gives a criteria on the existence of minimizers for (1.2) based on the value of $\mathcal{O}(b_1, b_2, \beta)$.

Proposition 2.1. *Suppose that (1.6) holds. Let b_1, b_2 and β be positive. Then*

(i) *(1.2) has at least one minimizer if $\mathcal{O}(b_1, b_2, \beta) > 1$.*

(ii) *(1.2) has no minimizer if $\mathcal{O}(b_1, b_2, \beta) < 1$.*

To establish Proposition 2.1, we need the following compactness lemma.

Lemma 2.1. *Suppose $V_i \in L^\infty_{loc}(\mathbb{R}^2)$ satisfies $\lim_{|x| \to \infty} V_i(x) = \infty$, where $i = 1, 2$. Then the embedding $\mathcal{X} = \mathcal{H}_1 \times \mathcal{H}_2 \hookrightarrow L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2)$ is compact for all $2 \leq q < \infty$.*

Since Lemma 2.1 can be proved in a similar way to that of [32, Theorem XIII.67] or [6, Theorem 2.1], we omit the proof. □

Lemma 2.2. *Let $\mathcal{O}(\cdot)$ be defined by (2.1), then $\mathcal{O}(\cdot)$ is locally Lipschitz continuous in $\mathbb{R}^3_+$.*

Proof. We first prove that

$$\frac{a^*}{\max\{b_1 + \beta, b_2 + \beta\}} \leq \mathcal{O}(b_1, b_2, \beta) \leq \frac{2a^*}{b_1 + b_2 + 2\beta}. \tag{2.2}$$

Indeed, the upper bound of (2.2) follows directly by taking $(\frac{Q(x)}{\|Q\|_2}, \frac{Q(x)}{\|Q\|_2})$ as a trial function of (2.1). On the other hand, for any $(u_1, u_2) \in H^1(\mathbb{R}^2) \times H^1(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} |u_i|^2 dx =$

$1, i = 1,\ 2$, it then follows from the Gagliardo-Nirenberg inequality (1.10) that

$$
\begin{aligned}
&\frac{\int_{\mathbb{R}^2}(|\nabla u_1|^2+|\nabla u_2|^2)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|u_1|^4dx+\frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4dx+\beta\int_{\mathbb{R}^2}|u_1|^2|u_2|^2dx}\\
\geq&\frac{\frac{a^*}{2}\int_{\mathbb{R}^2}(|u_1|^4+|u_1|^4)dx}{\frac{b_1+\beta}{2}\int_{\mathbb{R}^2}|u_1|^4dx+\frac{b_2+\beta}{2}\int_{\mathbb{R}^2}|u_2|^4dx}\geq\frac{a^*}{\max\{b_1+\beta,b_2+\beta\}},
\end{aligned}
$$

which then gives the lower bound of (2.2). Therefore, (2.2) is proved.

Consider $(b_1,b_2,\beta),(\tilde b_1,\tilde b_2,\tilde\beta)\in\mathbb{R}^3_+$, and let $\{(u_{1n},u_{2n})\}$ be a minimizing sequence of $\mathcal{O}(b_1,b_2,\beta)$. Since (2.1) is invariant under the rescaling: $u(x)\mapsto\lambda u(\lambda x),\lambda>0$, one may assume that

$$
\int_{\mathbb{R}^2}(|\nabla u_{1n}|^2+|\nabla u_{2n}|^2)dx=1\ \text{ for all } n\in\mathbb{N}^+. \tag{2.3}
$$

We then obtain that

$$
\int_{\mathbb{R}^2}|u_{in}|^4dx\leq\frac{2}{a^*}\int_{\mathbb{R}^2}|\nabla u_{in}|^2\leq\frac{2}{a^*},\ \ i=1,\ 2,
$$

and

$$
\begin{aligned}
\int_{\mathbb{R}^2}|u_{1n}|^2|u_{2n}|^2dx&\leq\frac12\int_{\mathbb{R}^2}(|u_{1n}|^4+|u_{2n}|^4)dx\\
&\leq\frac{1}{a^*}\int_{\mathbb{R}^2}\big(|\nabla u_{1n}|^2+|\nabla u_{2n}|^2\big)dx=\frac{1}{a^*}.
\end{aligned}
$$

Furthermore, we have

$$
\begin{aligned}
\frac{1}{\mathcal{O}(b_1,b_2,\beta)}=&\lim_{n\to\infty}\Big[\frac{\frac{\tilde b_1}{2}\int_{\mathbb{R}^2}|u_{1n}|^4dx+\frac{\tilde b_2}{2}\int_{\mathbb{R}^2}|u_{2n}|^4dx+\tilde\beta\int_{\mathbb{R}^2}|u_{1n}|^2|u_{2n}|^2dx}{\int_{\mathbb{R}^2}(|\nabla u_{1n}|^2+|\nabla u_{2n}|^2)dx}\\
&+\sum_{i=1}^2\frac{b_i-\tilde b_i}{2}\int_{\mathbb{R}^2}|u_{in}|^4dx+(\beta-\tilde\beta)\int_{\mathbb{R}^2}|u_{1n}|^2|u_{2n}|^2dx\Big]\\
\leq&\lim_{n\to\infty}\Big[\frac{1}{\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}+\sum_{i=1}^2\frac{|b_i-\tilde b_i|}{2}\int_{\mathbb{R}^2}|u_{in}|^4dx+|\beta-\tilde\beta|\int_{\mathbb{R}^2}|u_{1n}|^2|u_{2n}|^2dx\Big]\\
\leq&\frac{1}{\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}+\sum_{i=1}^2\frac{|b_i-\tilde b_i|}{a^*}+\frac{|\beta-\tilde\beta|}{a^*},
\end{aligned}
$$

i.e.,

$$
\frac{1}{\mathcal{O}(b_1,b_2,\beta)}-\frac{1}{\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}\leq\frac{3}{a^*}\big|(b_1,b_2,\beta)-(\tilde b_1,\tilde b_2,\tilde\beta)\big|. \tag{2.4}
$$

Similarly, taking $\{(\tilde u_{1n},\tilde u_{2n})\}$ as a minimizing sequence of $\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)$, and repeating the above argument, we know that

$$
\frac{1}{\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}-\frac{1}{\mathcal{O}(b_1,b_2,\beta)}\leq\frac{3}{a^*}\big|(b_1,b_2,\beta)-(\tilde b_1,\tilde b_2,\tilde\beta)\big|,
$$

The above estimates then yield that

$$\Big|\frac{\mathcal{O}(b_1,b_2,\beta)-\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}{\mathcal{O}(b_1,b_2,\beta)\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}\Big| = \Big|\frac{1}{\mathcal{O}(b_1,b_2,\beta)}-\frac{1}{\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)}\Big|$$
$$\le \frac{3}{a^*}\big|(b_1,b_2,\beta)-(\tilde b_1,\tilde b_2,\tilde\beta)\big|.$$

By applying (2.2), we therefore conclude that

$$\big|\mathcal{O}(b_1,b_2,\beta)-\mathcal{O}(\tilde b_1,\tilde b_2,\tilde\beta)\big| \le \frac{12a^*}{(b_1+b_2+2\beta)(\tilde b_1+\tilde b_2+2\tilde\beta)}\big|(b_1,b_2,\beta)-(\tilde b_1,\tilde b_2,\tilde\beta)\big|,$$

which implies that $\mathcal{O}(\cdot)$ is locally Lipschitz continuous in $\mathbb{R}^3_+$. □

With Lemma 2.2, we now prove Proposition 2.1.

Proof of Proposition 2.1. i) Let $\{(u_{1n},u_{2n})\}\subset\mathcal{X}$ be a minimizing sequence of problem (1.2), *i.e.*,

$$\|u_{1n}\|_2^2=\|u_{2n}\|_2^2=1 \quad\text{and}\quad \lim_{n\to\infty}E_{b_1,b_2,\beta}(u_{1n},u_{2n})=\hat e(b_1,b_2,\beta).$$

It then follows from (2.1) that

$$E_{b_1,b_2,\beta}(u_{1n},u_{2n})\ge\sum_{i=1}^2\int_{\mathbb{R}^2}\Big[\big(1-\frac{1}{\mathcal{O}(b_1,b_2,\beta)}\big)|\nabla u_{in}|^2+V_i(x)|u_{in}|^2\Big]dx. \tag{2.5}$$

If $\mathcal{O}(b_1,b_2,\beta)>1$, (2.5) implies that $\{(u_{1n},u_{2n})\}\subset\mathcal{X}$ is bounded in n. Thus, by the compactness of Lemma 2.1, there exist a subsequence of $\{(u_{1n},u_{2n})\}$ and $(u_1,u_2)\in\mathcal{X}$ such that

$$(u_{1n},u_{2n})\overset{n}{\rightharpoonup}(u_1,u_2)\quad\text{weakly in }\mathcal{X},$$
$$(u_{1n},u_{2n})\overset{n}{\to}(u_1,u_2)\quad\text{strongly in }L^q(\mathbb{R}^2)\times L^q(\mathbb{R}^2)\text{ for all }q\in[2,\infty).$$

Therefore, $\|u_1\|_2^2=\|u_2\|_2^2=1$ and $E_{b_1,b_2,\beta}(u_1,u_2)=\hat e(b_1,b_2,\beta)$. This proves part one of the proposition.

ii) Suppose now that $\mathcal{O}(b_1,b_2,\beta)<1$. One then may choose $(u_1,u_2)\in\mathcal{M}$ such that each u_i $(i=1,2)$ has compact support in $\mathbb{R}^2$ and satisfies

$$\frac{\int_{\mathbb{R}^2}(|\nabla u_1|^2+|\nabla u_2|^2)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|u_1|^4dx+\frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4dx+\beta\int_{\mathbb{R}^2}|u_1|^2|u_2|^2dx}\le\delta:=\frac{1+\mathcal{O}(b_1,b_2,\beta)}{2}<1. \tag{2.6}$$

For $\lambda>0$, define

$$\bar u_i(x)=\lambda u_i(\lambda x),\ \ i=1,\,2, \tag{2.7}$$

so that $(\bar u_1,\bar u_2)\in\mathcal{M}$. Since $u_i(x)$ is compactly supported in $\mathbb{R}^2$ and $V_i(x)\in L^\infty_{\rm loc}(\mathbb{R}^2)$, there exists a positive constant C, independent of $\lambda>0$, such that for $\lambda\to\infty$,

$$\int_{\mathbb{R}^2}V_i(x)|\bar u_i|^2dx=\int_{\mathbb{R}^2}V_i\big(\frac{x}{\lambda}\big)|u_i|^2dx\le C<\infty,\ \ i=1,\,2. \tag{2.8}$$

On the other hand, by (2.6) and (2.7), we have

$$
\begin{aligned}
&\sum_{i=1}^{2}\int_{\mathbb{R}^2}\Big(|\nabla \bar u_i|^2-\frac{b_i}{2}|\bar u_i|^4\Big)dx-\beta\int_{\mathbb{R}^2}|\bar u_1|^2|\bar u_2|^2dx\\
&=\lambda^2\sum_{i=1}^{2}\int_{\mathbb{R}^2}\Big(|\nabla u_i|^2-\frac{b_i}{2}|u_i|^4\Big)dx-\beta\lambda^2\int_{\mathbb{R}^2}|u_1|^2|u_2|^2dx\\
&\le\lambda^2(\delta-1)\Big(\frac{b_1}{2}\int_{\mathbb{R}^2}|u_1|^4dx+\frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4dx+\beta\int_{\mathbb{R}^2}|u_1|^2|u_2|^2dx\Big)\\
&\to-\infty\quad\text{as}\quad\lambda\to\infty.
\end{aligned}
\tag{2.9}
$$

It then follows from (2.8) and (2.9) that

$$\hat e(b_1,b_2,\beta)\le E_{b_1,b_2,\beta}(\bar u_1,\bar u_2)\to-\infty \text{ as } \lambda\to\infty,$$

which implies that $\hat e(b_1,b_2,\beta)$ does not admit any minimizer. Proposition 2.1 is therefore established. □

We end this section by proving Theorem 1.1.

Proof of Theorem 1.1. (i): For any $(u_1,u_2)\in H^1(\mathbb{R}^2)\times H^1(\mathbb{R}^2)$ with $\int_{\mathbb{R}^2}|u_i|^2dx=1$, $i=1,\,2$, it follows from (1.10) that

$$
\begin{aligned}
&\mathcal{O}(b_1,b_2,\beta)\\
\ge&\inf_{\{\|u_i\|_2^2=1,i=1,2\}}\frac{\frac{a^*}{2}\int_{\mathbb{R}^2}(|u_1|^4+|u_2|^4)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|u_1|^4dx+\frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4dx+\beta\big(\int_{\mathbb{R}^2}|u_1|^4dx\int_{\mathbb{R}^2}|u_2|^4dx\big)^{\frac12}}\\
=&\inf_{\{\|u_i\|_2^2=1,i=1,2\}}\frac{\frac{a^*}{2}\Big(1+\int_{\mathbb{R}^2}|u_2|^4dx/\int_{\mathbb{R}^2}|u_1|^4dx\Big)}{\frac{b_1}{2}+\frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4dx/\int_{\mathbb{R}^2}|u_1|^4dx+\beta\big(\int_{\mathbb{R}^2}|u_2|^4dx/\int_{\mathbb{R}^2}|u_1|^4dx\big)^{\frac12}}.
\end{aligned}
$$

Setting

$$t:=\Big(\int_{\mathbb{R}^2}|u_2|^4dx/\int_{\mathbb{R}^2}|u_1|^4dx\Big)^{\frac12}\in(0,\infty)\text{ and } f_{b_1,b_2,\beta}(t):=\frac{\frac{a^*}{2}(1+t^2)}{\frac{b_1}{2}+\frac{b_2}{2}t^2+\beta t},\tag{2.10}$$

and then

$$\mathcal{O}(b_1,b_2,\beta)\ge\inf_{t\in(0,\infty)}f_{b_1,b_2,\beta}(t).\tag{2.11}$$

Since $0<b_i<a^*$ $(i=1,\,2)$ and $\beta<\sqrt{(a^*-b_1)(a^*-b_2)}$, standard calculations show that $f_{b_1,b_2,\beta}(t)>1$ for any $t\in(0,\infty)$, and also

$$\lim_{t\to0^+}f_{b_1,b_2,\beta}(t)=\frac{a^*}{b_1}>1\quad\text{and}\quad\lim_{t\to\infty}f_{b_1,b_2,\beta}(t)=\frac{a^*}{b_2}>1.$$

So, by the continuity of $f_{b_1,b_2,\beta}(t)$ we obtain that $\inf\limits_{t\in(0,\infty)}f_{b_1,b_2,\beta}(t)>1$. This estimate and (2.11) then imply that $\mathcal{O}(b_1,b_2,\beta)>1$, from which we conclude that (1.2) has at least one minimizer by Proposition 2.1(i).

(ii): Consider a function $0 \le \varphi \in C_0^\infty(\mathbb{R}^2)$ satisfying $\int_{\mathbb{R}^2} |\varphi|^2 dx = 1$, and set

$$u_\lambda(x) = \frac{\lambda}{\|Q\|_2} Q(\lambda x), \ \ \lambda > 0, \tag{2.12}$$

where $Q(x)$ is the unique radial positive solution of the scalar field equation (1.9). It then follows from (1.11) that

$$\int_{\mathbb{R}^2} |\nabla u_\lambda|^2 dx = \frac{\lambda^2}{\|Q\|_2^2} \int_{\mathbb{R}^2} |\nabla Q|^2 dx = \lambda^2,$$

as well as

$$\int_{\mathbb{R}^2} |u_\lambda|^4 dx = \frac{\lambda^2}{\|Q\|_2^4} \int_{\mathbb{R}^2} |Q|^4 dx = \frac{2\lambda^2}{a^*}.$$

Suppose now that $b_1 > a^*$. We then take (u_λ, φ) as a trial function of $\mathcal{O}$ so that

$$\begin{aligned}
\mathcal{O}(b_1, b_2, \beta) &\le \frac{\int_{\mathbb{R}^2} \big(|\nabla u_\lambda|^2 + |\nabla \varphi|^2\big) dx}{\frac{b_1}{2} \int_{\mathbb{R}^2} |u_\lambda|^4 dx + \frac{b_2}{2} \int_{\mathbb{R}^2} |\varphi|^4 dx + \beta \int_{\mathbb{R}^2} |u_\lambda|^2 |\varphi|^2 dx} \\
&= \frac{\lambda^2 + \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx}{\frac{b_1}{a^*}\lambda^2 + \frac{b_2}{2} \int_{\mathbb{R}^2} |\varphi|^4 dx + \frac{\beta}{a^*} \int_{\mathbb{R}^2} |Q|^2 |\varphi(\frac{x}{\lambda})|^2 dx} \\
&\le \frac{\lambda^2 + \int_{\mathbb{R}^2} |\nabla \varphi|^2 dx}{\frac{b_1}{a^*}\lambda^2} \to \frac{a^*}{b_1} < 1 \quad \text{as} \quad \lambda \to \infty.
\end{aligned}$$

Thus, $\mathcal{O}(b_1, b_2, \beta) < 1$ and it then follows from Proposition 2.1(ii) that (1.2) has no minimizer. Similarly, if $b_2 > a^*$, one can also obtain the nonexistence of minimizers for (1.2).

Assume finally that $\beta > \frac{a^* - b_1}{2} + \frac{a^* - b_2}{2}$. In this case, take (u_λ, u_λ) as a trial function of $\mathcal{O}$, where $u_\lambda \ge 0$ is defined by (2.12). We then have

$$\mathcal{O}(b_1, b_2, \beta) \le \frac{2\int_{\mathbb{R}^2} |\nabla u_\lambda|^2 dx}{\big(\frac{b_1}{2} + \frac{b_2}{2} + \beta\big) \int_{\mathbb{R}^2} |u_\lambda|^4 dx} = \frac{a^*}{\frac{b_1}{2} + \frac{b_2}{2} + \beta} < 1.$$

Hence, it follows again from Proposition 2.1(ii) that (1.2) has no minimizer. This completes the proof of Theorem 1.1. □

3 Further results on the existence of minimizers

As discussed in the Introduction, our Theorem 1.1 gives a complete classification of the existence of minimizers for (1.2), except that (b_1, b_2, β) satisfies

$$0 < b_1 \le a^*, \ \ 0 < b_2 \le a^* \text{ and } \beta \in \Big[\sqrt{(a^* - b_1)(a^* - b_2)}, \frac{a^* - b_1}{2} + \frac{a^* - b_2}{2}\Big].$$

The aim of this section is to prove Theorems 1.2 and 1.3, which are concerned with the existence of minimizers for (1.2) when (b_1, b_2, β) lies in the above range. It turn out that such an existence depends on whether $0 < b_1 = b_2 \le a^*$ or not.

Firstly, we shall make full use of Proposition 2.1 to derive the existence of minimizers for (1.2) in the case where $0 < b_1 \ne b_2 < a^*$ and β is close to $\sqrt{(a^* - b_1)(a^* - b_2)}$ from above. We start with the following lemma.

Lemma 3.1. *Let $0 < b_1 \neq b_2 < a^*$ and $\beta = \sqrt{(a^* - b_1)(a^* - b_2)}$ such that*

$$|b_1 - b_2| \leq 2\beta.$$

If $\mathcal{O}\big(b_1, b_2, \sqrt{(a^* - b_1)(a^* - b_2)}\big)$ possesses a radially symmetric (about the origin) minimizing sequence $\{(u_{1n}, u_{2n})\} \subset H_r^1(\mathbb{R}^2) \times H_r^1(\mathbb{R}^2)$ satisfying

$$0 < C_1 \leq \int_{\mathbb{R}^2} |\nabla u_{in}|^2 dx \leq C_2 < \infty, \text{ and } 0 < C_1 \leq \int_{\mathbb{R}^2} |u_{in}|^4 dx \leq C_2 < \infty, \ \ i = 1, 2, \tag{3.1}$$

and C_1, C_2 are independent of n, then we have

$$\mathcal{O}\big(b_1, b_2, \sqrt{(a^* - b_1)(a^* - b_2)}\big) > 1.$$

Proof. Since $\{u_{in} \subset H_r^1(\mathbb{R}^2)\}$ $(i = 1, 2)$ is radially symmetric, by (3.1) and the compactness lemma of Strauss [34], we deduce that there exists $u_i(x) \in H_r^1(\mathbb{R}^2)$ satisfying

$$\begin{aligned} &u_{in} \overset{n}{\rightharpoonup} u_i \quad \text{weakly in} \quad H^1(\mathbb{R}^2), \ \ i = 1,\ 2; \\ &u_{in} \overset{n}{\to} u_i \quad \text{strongly in} \quad L^p(\mathbb{R}^2), \ \ \forall\, p \in (2, \infty), \ \ i = 1,\ 2. \end{aligned} \tag{3.2}$$

Also, the assumption (3.1) implies that

$$\int_{\mathbb{R}^2} |u_i|^4 dx \geq C_1 > 0 \text{ and } u_i \not\equiv 0, \ i = 1,\ 2.$$

Since $0 < b_1 \neq b_2 < a^*$, without loss of generality, we may assume that $b_1 < b_2$. Then the assumption on β can be simplified as

$$0 < b_1 < b_2 < a^* \quad \text{and} \quad b_2 \leq 2\beta + b_1, \text{ where } \ \beta = \sqrt{(a^* - b_1)(a^* - b_2)}. \tag{3.3}$$

Applying (1.10) and (3.2), we have

$$\begin{aligned} \mathcal{O}(b_1, b_2, \beta) &\geq \lim_{n\to\infty} \frac{\frac{a^*}{2}\int_{\mathbb{R}^2}(|u_{1n}|^4 + |u_{2n}|^4)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|u_{1n}|^4 dx + \frac{b_2}{2}\int_{\mathbb{R}^2}|u_{2n}|^4 dx + \beta\int_{\mathbb{R}^2}|u_{1n}|^2|u_{2n}|^2 dx} \\ &= \frac{\frac{a^*}{2}\int_{\mathbb{R}^2}(|u_1|^4 + |u_2|^4)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|u_1|^4 dx + \frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4 dx + \beta\int_{\mathbb{R}^2}|u_1|^2|u_2|^2 dx} \\ &\geq f_{b_1,b_2,\beta}(t_0), \ \ t_0 := \Big(\int_{\mathbb{R}^2}|u_2|^4 dx \Big/ \int_{\mathbb{R}^2}|u_1|^4 dx\Big)^{\frac{1}{2}} \in (0, \infty), \end{aligned} \tag{3.4}$$

where $f_{b_1,b_2,\beta}(t)$ is defined in (2.10), and the equality of (3.4) holds if and only if

$$u_2^2(x) = \kappa u_1^2(x) \quad \text{for some} \quad \kappa > 0. \tag{3.5}$$

Moreover, since $\beta = \sqrt{(a^* - b_1)(a^* - b_2)}$, it holds that

$$f_{b_1,b_2,\beta}(t) \geq 1, \ \ \forall\, t \in (0, \infty), \tag{3.6}$$

and

$$f_{b_1,b_2,\beta}(t) = 1 \quad \Leftrightarrow \quad t = t_1 := \sqrt{\frac{a^* - b_1}{a^* - b_2}}. \tag{3.7}$$

We thus deduce from (3.4) and (3.6) that $\mathcal{O}\big(b_1, b_2, \sqrt{(a^*-b_1)(a^*-b_2)}\big) \geq 1$.

We claim that $\mathcal{O}\big(b_1, b_2, \sqrt{(a^*-b_1)(a^*-b_2)}\big) > 1$. Otherwise, if

$$\mathcal{O}\big(b_1, b_2, \sqrt{(a^*-b_1)(a^*-b_2)}\big) = 1, \tag{3.8}$$

then (3.5) holds for $t_0 = t_1$, where t_0 and t_1 are given by (3.4) and (3.7), respectively. Thus, we have

$$u_2^2(x) = \kappa u_1^2(x), \text{ where } \kappa = t_0 = \sqrt{\frac{a^*-b_1}{a^*-b_2}} > 1. \tag{3.9}$$

Together with (3.2), this implies that

$$\begin{aligned}\mathcal{O}(b_1, b_2, \beta) &\geq \frac{\int_{\mathbb{R}^2}(|\nabla u_1|^2 + |\nabla u_2|^2)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|u_1|^4dx + \frac{b_2}{2}\int_{\mathbb{R}^2}|u_2|^4dx + \beta\int_{\mathbb{R}^2}|u_1|^2|u_2|^2dx}\\ &= \frac{1+\kappa}{\frac{b_1}{2} + \frac{b_2}{2}\kappa^2 + \beta\kappa}\cdot\frac{\int_{\mathbb{R}^2}|\nabla u_1|^2dx}{\int_{\mathbb{R}^2}|u_1|^4dx}.\end{aligned} \tag{3.10}$$

On the other hand, define

$$\tilde{u}_i(x) = \frac{1}{\sqrt{\lambda_i}}u_i(x) \text{ where } \lambda_i := \int_{\mathbb{R}^2}|u_i|^2dx \leq \lim_{n\to\infty}\int_{\mathbb{R}^2}|u_{in}|^2dx = 1, \tag{3.11}$$

so that $\int_{\mathbb{R}^2}|\tilde{u}_i|^2dx = 1$, where $i = 1, 2$. Note also from (3.9) that

$$\lambda_2 = \kappa\lambda_1 \leq 1. \tag{3.12}$$

Therefore, by the definition of $\mathcal{O}(\cdot)$, we deduce from (3.9), (3.11) and (3.12) that

$$\begin{aligned}\mathcal{O}(b_1, b_2, \beta) &\leq \frac{\int_{\mathbb{R}^2}(|\nabla \tilde{u}_1|^2 + |\nabla \tilde{u}_2|^2)dx}{\frac{b_1}{2}\int_{\mathbb{R}^2}|\tilde{u}_1|^4dx + \frac{b_2}{2}\int_{\mathbb{R}^2}|\tilde{u}_2|^4dx + \beta\int_{\mathbb{R}^2}|\tilde{u}_1|^2|\tilde{u}_2|^2dx}\\ &= \frac{2\lambda_1}{\frac{b_1}{2} + \frac{b_2}{2} + \beta}\cdot\frac{\int_{\mathbb{R}^2}|\nabla u_1|^2dx}{\int_{\mathbb{R}^2}|u_1|^4dx} \leq \frac{\frac{2}{\kappa}}{\frac{b_1}{2} + \frac{b_2}{2} + \beta}\cdot\frac{\int_{\mathbb{R}^2}|\nabla u_1|^2dx}{\int_{\mathbb{R}^2}|u_1|^4dx}.\end{aligned}$$

It then follows from (3.10) that

$$\frac{1+\kappa}{\frac{b_1}{2} + \frac{b_2}{2}\kappa^2 + \beta\kappa} \leq \frac{\frac{2}{\kappa}}{\frac{b_1}{2} + \frac{b_2}{2} + \beta},$$

i.e.,

$$\frac{\kappa+\kappa^2}{\frac{b_1}{2} + \frac{b_2}{2}\kappa^2 + \beta\kappa} \leq \frac{2}{\frac{b_1}{2} + \frac{b_2}{2} + \beta}, \text{ where } \kappa = \sqrt{\frac{a^*-b_1}{a^*-b_2}} > 1,$$

which however contradicts Lemma A.1 in the Appendix. Thus (3.8) cannot occur, then $\mathcal{O}\big(b_1, b_2, \sqrt{(a^*-b_1)(a^*-b_2)}\big) > 1$. □

With Lemma 3.1 and Proposition 2.1, we can prove now Theorem 1.2.

Proof of Theorem 1.2. Let $\{(u_{1n}, u_{2n})\}$ be a minimizing sequence of $\mathcal{O}(b_1, b_2, \beta)$. By the Schwarz symmetrization of $\{(u_{1n}, u_{2n})\}$, one may assume that

$$u_{in}(x) = u_{in}(|x|) \geq 0, \ \ i = 1, 2. \tag{3.13}$$

Moreover, since the problem (2.1) is invariant under the rescaling: $u(x) \mapsto \lambda u(\lambda x)$ for $\lambda > 0$, one can also assume that

$$\int_{\mathbb{R}^2} (|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2) dx = 1 \quad \text{for all } n \in \mathbb{N}^+. \tag{3.14}$$

By the Gagliardo-Nirenberg inequality (1.10), we then obtain that $\int_{\mathbb{R}^2} |u_{in}|^4 dx$ ($i = 1, 2$) is bounded uniformly, *i.e.*,

$$0 < C_1 \le \frac{b_1}{2} \int_{\mathbb{R}^2} |u_{1n}|^4 dx + \frac{b_2}{2} \int_{\mathbb{R}^2} |u_{2n}|^4 dx + \beta \int_{\mathbb{R}^2} |u_{1n}|^2 |u_{2n}|^2 dx \le C_2 < \infty. \tag{3.15}$$

Under the assumption (1.15), we claim that

$$\mathcal{O}\big(b_1, b_2, \sqrt{(a^* - b_1)(a^* - b_2)}\big) > 1. \tag{3.16}$$

We prove (3.16) by considering separately the following two cases.

Case 1. If

$$\int_{\mathbb{R}^2} |u_{in}|^4 dx \to 0 \quad \text{as} \quad n \to \infty, \quad \text{where} \quad i = 1 \text{ or } 2. \tag{3.17}$$

Without loss of generality, we assume that

$$\int_{\mathbb{R}^2} |u_{1n}|^4 dx \to 0 \quad \text{as} \quad n \to \infty.$$

It then follows from (3.15) that

$$\int_{\mathbb{R}^2} |u_{1n}|^2 |u_{2n}|^2 dx \xrightarrow{n} 0 \quad \text{and} \quad \int_{\mathbb{R}^2} |u_{2n}|^4 dx \ge C > 0.$$

By the assumption $0 < b_2 < a^*$, we then have

$$\begin{aligned}
\mathcal{O}(b_1, b_2, \beta) &= \lim_{n\to\infty} \frac{\int_{\mathbb{R}^2} \big(|\nabla u_{1n}|^2 + |\nabla u_{2n}|^2\big) dx}{\frac{b_2}{2} \int_{\mathbb{R}^2} |u_{2n}|^4 dx + o(1)} \\
&\ge \lim_{n\to\infty} \frac{\int_{\mathbb{R}^2} |\nabla u_{2n}|^2 dx}{\frac{b_2}{2} \int_{\mathbb{R}^2} |u_{2n}|^4 dx + o(1)} \ge \frac{a^*}{b_2} > 1,
\end{aligned} \tag{3.18}$$

where (1.10) is used. Thus, (3.16) follows immediately from (3.18) with $\beta = \sqrt{(a^* - b_1)(a^* - b_2)}$.

Case 2. If

$$\int_{\mathbb{R}^2} |u_{in}|^4 dx \ge C > 0 \quad \text{for} \quad i = 1 \text{ and } 2. \tag{3.19}$$

In this case, applying (1.10) and (3.14), we deduce that there exist positive constants C_3 and C_4, independent of n, such that

$$\begin{aligned}
0 < &C_3 \le \int_{\mathbb{R}^2} |\nabla u_{in}|^2 dx \le C_4 < \infty, \;\; i = 1, 2, \\
0 < &C_3 \le \int_{\mathbb{R}^2} |u_{in}|^4 dx \le C_4 < \infty, \quad\; i = 1, 2.
\end{aligned}$$

Under the assumption (1.15), we then conclude from (3.13) and Lemma 3.1 that the estimate (3.16) also holds. So, the claim is proved.

Combining (3.16) and Lemma 2.2, we then derive that there exists a constant δ satisfying $\delta > \sqrt{(a^* - b_1)(a^* - b_2)}$ such that

$$\mathcal{O}(b_1, b_2, \beta) > 1 \quad \text{for all} \quad \beta \in \big[\sqrt{(a^* - b_1)(a^* - b_2)}, \delta\big).$$

By Proposition 2.1 we therefore conclude that (1.2) has at least one minimizer. $\square$

Next, we turn to proving Theorem 1.3, that is, the existence and non-existence of minimizers for (1.2) in the case where (b_1, b_2, β) satisfies $0 < b_1 = b_2 < a^*$ and (1.14). These assumptions on (b_1, b_2, β) imply that (b_1, b_2, β) lies on the segment: $(b_1, b_2, \beta) = (a^* - \beta, a^* - \beta, \beta)$ with $0 < \beta < a^*$. In this case, for simplicity, we rewrite the functional (1.5) as

$$\begin{aligned} E_{a^*,a^*}(u_1, u_2) := & \sum_{i=1}^{2} \int_{\mathbb{R}^2} \Big(|\nabla u_i|^2 + V_i(x)|u_i|^2 - \frac{a^*}{2}|u_i|^4 \Big) dx \\ & + \frac{\beta}{2} \int_{\mathbb{R}^2} \big(|u_1|^2 - |u_2|^2 \big)^2 dx \,, \quad \text{where} \quad (u_1, u_2) \in \mathcal{X} \,. \end{aligned} \tag{3.20}$$

Note from (A.5) and (A.6) in the Appendix that

$$0 \le \hat{e}(a^* - \beta, a^* - \beta, \beta) \le \inf_{x \in \mathbb{R}^2} \big(V_1(x) + V_2(x) \big).$$

If $\inf_{x \in \mathbb{R}^2} \big(V_1(x) + V_2(x) \big) = 0$ is attained, Theorem 1.3(i) shows that (1.2) does not admit any minimizer. Surprisingly, Theorem 1.3(ii) however shows that there exists at least one minimizer for (1.2) if

$$0 \le \hat{e}(a^* - \beta, a^* - \beta, \beta) < \inf_{x \in \mathbb{R}^2} \big(V_1(x) + V_2(x) \big).$$

As illustrated in Example 1.1, the above condition does hold for some potentials $V_1(x)$ and $V_2(x)$.

Finally, we give the proof of Theorem 1.3.

Proof of Theorem 1.3 (i): In order to prove part (i), we assume that $\inf_{x \in \mathbb{R}^2} \big(V_1(x) + V_2(x) \big) = 0$, i.e., there exists $x_0 \in \mathbb{R}^2$ such that $V_1(x_0) = V_2(x_0) = 0$. Since $b_1 = b_2 = a^* - \beta$, we take $\phi > 0$ as in (A.3) and use (ϕ, ϕ) with $\bar{x}_0 = x_0$ as a trial function of $\hat{e}(a^* - \beta, a^* - \beta, \beta)$. It then follows from (A.5) and (A.6) that

$$0 \le \hat{e}(a^* - \beta, a^* - \beta, \beta) \le \lim_{\tau \to \infty} E_{a^*,a^*}(\phi, \phi) = V_1(x_0) + V_2(x_0) = 0, \tag{3.21}$$

i.e., $\hat{e}(a^* - \beta, a^* - \beta, \beta) = 0$. Suppose now there exists a minimizer $(\hat{u}_1, \hat{u}_2) \in \mathcal{M}$ for $\hat{e}(a^* - \beta, a^* - \beta, \beta)$. As pointed out in the Introduction, we can assume $(\hat{u}_1, \hat{u}_2)$ to be non-negative. It then follows from (3.20) that $\hat{u}_1 \equiv \hat{u}_2 \ge 0$ in $\mathbb{R}^2$, and

$$\int_{\mathbb{R}^2} |\nabla \hat{u}_1|^2 dx = \frac{1}{2} \int_{\mathbb{R}^2} |\hat{u}_1|^4 dx \;\; \text{and} \;\; \int_{\mathbb{R}^2} V_1(x) |\hat{u}_1|^2 dx = 0.$$

This is a contradiction, since the first equality implies that $\hat{u}_1(x)$ is equal to (up to translation) $Q(x)$, but the second equality yields that $\hat{u}_1(x)$ has compact support. Therefore, the conclusion (i) of Theorem 1.3 is proved.

(ii): In this case, we have

$$0 \le \hat{e}(a^* - \beta, a^* - \beta, \beta) < \inf_{x\in\mathbb{R}^2} \big(V_1(x) + V_2(x)\big).$$

For $\mathcal{M}$ defined by (1.3), we introduce

$$d(\vec{u}, \vec{v}) := \|\vec{u} - \vec{v}\|_{\mathcal{X}}, \quad \vec{u},\, \vec{v} \in \mathcal{M}\,,$$

where

$$\|\vec{u}\|_{\mathcal{X}} = \big(\|u_1\|^2_{\mathcal{H}_1} + \|u_2\|^2_{\mathcal{H}_2}\big)^{\frac{1}{2}}, \quad \vec{u} = (u_1, u_2) \in \mathcal{X}\,.$$

It is easy to check that $(\mathcal{M}, d)$ is a complete distance space. Hence, by Ekeland's variational principle [35, Theorem 5.1], there exists a minimizing sequence $\{\vec{u}_n = (u_{1n}, u_{2n})\} \subset \mathcal{M}$ of $\hat{e}(a^* - \beta, a^* - \beta, \beta)$ such that

$$\hat{e}(a^* - \beta, a^* - \beta, \beta) \le E_{a^*,a^*}(\vec{u}_n) \le \hat{e}(a^* - \beta, a^* - \beta, \beta) + \frac{1}{n}\,, \tag{3.22}$$

$$E_{a^*,a^*}(\vec{v}) \ge E_{a^*,a^*}(\vec{u}_n) - \frac{1}{n}\|\vec{u}_n - \vec{v}\|_{\mathcal{X}} \quad \text{for} \quad \vec{v} \in \mathcal{M}. \tag{3.23}$$

Due to the compactness of Lemma 2.1, in order to show that there exists a minimizer for $\hat{e}(a^* - \beta, a^* - \beta, \beta)$, it suffices to prove that $\{\vec{u}_n = (u_{1n}, u_{2n})\}$ is bounded in $\mathcal{X}$ uniformly w.r.t. n. We argue by contradiction. If $\{\vec{u}_n = (u_{1n}, u_{2n})\}$ is unbounded in $\mathcal{X}$, then there exists a subsequence of $\{\vec{u}_n\}$, still denoted by $\{\vec{u}_n\}$, such that $\|\vec{u}_n\|_{\mathcal{X}} \overset{n}{\to} \infty$. By Gagliardo-Nirenberg inequality, we deduce from (3.22) that

$$\sum_{i=1}^{2} \int_{\mathbb{R}^2} V_i(x)|u_{in}|^2 dx \le E_{a^*,a^*}(\vec{u}_n) \le \hat{e}(a^* - \beta, a^* - \beta, \beta) + \frac{1}{n}. \tag{3.24}$$

Hence,

$$\int_{\mathbb{R}^2} |\nabla u_{1n}|^2 + |\nabla u_{2n}|^2 dx \overset{n}{\to} \infty. \tag{3.25}$$

We now claim that

$$\int_{\mathbb{R}^2} |\nabla u_{in}|^2 dx \sim \frac{a^*}{2} \int_{\mathbb{R}^2} |u_{in}|^4 dx \overset{n}{\to} \infty, \quad i = 1,\, 2, \tag{3.26}$$

$$\int_{\mathbb{R}^2} |u_{1n}|^4 dx \Big/ \int_{\mathbb{R}^2} |u_{2n}|^4 dx \overset{n}{\to} 1. \tag{3.27}$$

Indeed, by (3.25), we may assume that $\int_{\mathbb{R}^2} |\nabla u_{1n}|^2 dx \overset{n}{\to} \infty$. Note from (3.22) that

$$0 \le \int_{\mathbb{R}^2} |\nabla u_{in}|^2 dx - \frac{a^*}{2} \int_{\mathbb{R}^2} |u_{in}|^4 dx \le \hat{e}(a^* - \beta, a^* - \beta, \beta) + \frac{1}{n}, \ \text{ for } \ i = 1,\, 2. \tag{3.28}$$

This implies that

$$\frac{a^*}{2} \int_{\mathbb{R}^2} |u_{1n}|^4 dx \overset{n}{\to} \infty \quad \text{and} \quad \int_{\mathbb{R}^2} |\nabla u_{1n}|^2 dx \Big/ \frac{a^*}{2} \int_{\mathbb{R}^2} |u_{1n}|^4 dx \overset{n}{\to} 1. \tag{3.29}$$

On the other hand, (3.22) also yields that

$$\beta\int_{\mathbb{R}^2}\big(|u_{1n}|^2-|u_{2n}|^2\big)^2dx\le \hat{e}(a^*-\beta,a^*-\beta,\beta)+\frac{1}{n}. \tag{3.30}$$

Then,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2}|u_{a_1}|^4dx-\int_{\mathbb{R}^2}|u_{a_2}|^4dx\Big|\le\Big(\int_{\mathbb{R}^2}\big(|u_{a_1}|^2-|u_{a_2}|^2\big)^2dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^2}\big(|u_{a_1}|^2+|u_{a_2}|^2\big)^2dx\Big)^{\frac12}\\&\le\Big\{\frac{1}{\beta}\big(\hat{e}(a^*-\beta,a^*-\beta,\beta)+\frac{1}{n}\big)\Big\}^{\frac12}\Big[\Big(\int_{\mathbb{R}^2}|u_{a_1}|^4dx\Big)^{\frac12}+\Big(\int_{\mathbb{R}^2}|u_{a_2}|^4dx\Big)^{\frac12}\Big]\end{aligned}$$

Together with (3.29), we thus derive that

$$\int_{\mathbb{R}^2}|u_{2n}|^4dx\overset{n}{\to}\infty\quad\text{and}\quad\int_{\mathbb{R}^2}|u_{1n}|^4dx\Big/\int_{\mathbb{R}^2}|u_{2n}|^4dx\overset{n}{\to}1. \tag{3.31}$$

This estimate and (3.28) then imply that

$$\int_{\mathbb{R}^2}|\nabla u_{2n}|^2dx\overset{n}{\to}\infty\quad\text{and}\quad\int_{\mathbb{R}^2}|\nabla u_{2n}|^2dx\Big/\frac{a^*}{2}\int_{\mathbb{R}^2}|u_{2n}|^4dx\overset{n}{\to}1. \tag{3.32}$$

Therefore, (3.26) and (3.27) follow from (3.29), (3.31) and (3.32).

Define now

$$\epsilon_n^{-2}:=\int_{\mathbb{R}^2}|u_{1n}|^4dx.$$

Similar to Lemma 5.3 (i) in Section 5, there exists a sequence $\{y_{\epsilon_n}\}\subset\mathbb{R}^2$ as well as positive constants R_0 and η such that

$$w_{in}(x):=\epsilon_n u_{in}(\epsilon_n x+\epsilon_n y_{\epsilon_n}),\ \ i=1,\,2, \tag{3.33}$$

satisfies

$$\int_{B_{R_0}(0)}|w_{in}|^2dx>\eta>0,\ \ i=1,\,2. \tag{3.34}$$

Recall from (3.30) that

$$\int_{\mathbb{R}^2}\big(|w_{1n}|^2-|w_{2n}|^2\big)^2dx=\epsilon_n^2\int_{\mathbb{R}^2}\big(|u_{1n}|^2-|u_{2n}|^2\big)^2dx\overset{n}{\to}0,$$

which implies that

$$w_{1n}^2-w_{2n}^2\overset{n}{\to}0\ \text{ in } L^2(\mathbb{R}^2)\ \text{ and }\ w_{1n}^2-w_{2n}^2\overset{n}{\to}0\ \text{ a.e. in }\mathbb{R}^2. \tag{3.35}$$

Moreover, since $\lim_{|x|\to\infty}V_i(x)=\infty$ $(i=1,2)$, it follows from (3.24) that

$$\sum_{i=1}^2\int_{\mathbb{R}^2}V_i(x)|u_{in}|^2dx=\sum_{i=1}^2\int_{\mathbb{R}^2}V_i(\epsilon_n x+\epsilon_n y_{\epsilon_n})|w_{in}|^2dx\le\hat{e}(a^*-\beta,a^*-\beta,\beta)+\frac{1}{n}. \tag{3.36}$$

We deduce from (3.34) and Fatou's Lemma that $\{\epsilon_n y_{\epsilon_n}\}$ is bounded uniformly in $\mathbb{R}^2$.

For any $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$, define

$$\tilde{\varphi}(x) = \varphi\Big(\frac{x - \epsilon_n y_{\epsilon_n}}{\epsilon_n}\Big), \ \ j(\tau, \sigma) = \frac{1}{2}\int_{\mathbb{R}^2} \big|u_{1n} + \tau u_{1n} + \sigma\tilde{\varphi}\big|^2 dx,$$

so that $j(\tau, \sigma)$ satisfies

$$j(0,0) = \frac{1}{2}, \ \ \frac{\partial j(0,0)}{\partial \tau} = \int_{\mathbb{R}^2} |u_{1n}|^2 dx = 1 \ \text{ and } \ \frac{\partial j(0,0)}{\partial \sigma} = \int_{\mathbb{R}^2} u_{1n}\tilde{\varphi} dx.$$

Applying the implicit function theorem then gives that there exist a constant $\delta_n > 0$ and a function $\tau(\sigma) \in C^1\big((-\delta_n, \delta_n), \mathbb{R}\big)$ such that

$$\tau(0) = 0, \ \ \tau'(0) = -\int_{\mathbb{R}^2} u_{1n}\tilde{\varphi} dx, \ \text{ and } \ j(\tau(\sigma), \sigma) = j(0,0) = \frac{1}{2}.$$

This implies that

$$\big(u_{1n} + \tau(\sigma) u_{1n} + \sigma\tilde{\varphi}, u_{2n}\big) \in \mathcal{M}, \ \text{ where } \ \sigma \in (-\delta_n, \delta_n).$$

We then obtain from (3.23) that

$$E_{a^*,a^*}(u_{1n} + \tau(\sigma)u_{1n} + \sigma\tilde{\varphi}, u_{2n}) - E_{a^*,a^*}(u_{1n}, u_{2n}) \geq -\frac{1}{n}\|(\tau(\sigma)u_{1n} + \sigma\tilde{\varphi}, 0)\|_{\mathcal{X}}.$$

Setting $\sigma \to 0^+$ and $\sigma \to 0^-$, respectively, we thus have

$$\Big|\big\langle E'_{a^*,a^*}(u_{1n}, u_{2n}), (\tau'(0)u_{1n} + \tilde{\varphi}, 0)\big\rangle\Big| \leq \frac{1}{n}\|\tau'(0)u_{1n} + \tilde{\varphi}\|_{\mathcal{H}_1}. \tag{3.37}$$

By the definition of (3.33), direct calculations yield that for $n \to \infty$,

$$\begin{aligned}\frac{1}{2}\big\langle E'_{a^*,a^*}(u_{1n}, u_{2n}), (\tilde{\varphi}, 0)\big\rangle = {} & \frac{1}{\epsilon_n}\int_{\mathbb{R}^2} \nabla w_{1n}\nabla\varphi dx + \epsilon_n\int_{\mathbb{R}^2} V_1(\epsilon_n x + \epsilon_n y_{\epsilon_n}) w_{1n}\varphi dx \\ & - \frac{a^*}{\epsilon_n}\int_{\mathbb{R}^2} w_{1n}^3\varphi dx + \frac{\beta}{\epsilon_n}\int_{\mathbb{R}^2} (w_{1n}^2 - w_{2n}^2) w_{1n}\varphi dx,\end{aligned} \tag{3.38}$$

and

$$\begin{aligned}&\tau'(0) = -\int_{\mathbb{R}^2} u_{1n}\tilde{\varphi} dx = -\epsilon_n\int_{\mathbb{R}^2} w_{1n}\varphi dx\,, \quad \|\tau'(0)u_{1n} + \tilde{\varphi}\|_{\mathcal{H}_1} < C, \\ &\mu_{1n} := \frac{1}{2}\langle E'_{a^*,a^*}(u_{1n}, u_{2n}), (u_{1n}, 0)\rangle \sim -\frac{a^*}{2}\int_{\mathbb{R}^2} |u_{1n}|^4 dx = -\frac{a^*}{2}\epsilon_n^{-2}.\end{aligned} \tag{3.39}$$

We then deduce from (3.37)-(3.39) that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^2} \nabla w_{1n}\nabla\varphi dx + \epsilon_n^2\int_{\mathbb{R}^2} V_1(\epsilon_n x + \epsilon_n y_{\epsilon_n}) w_{1n}\varphi dx - \mu_{1n}\epsilon_n^2\int_{\mathbb{R}^2} w_{1n}\varphi dx \\ &- a^*\int_{\mathbb{R}^2} w_{1n}^3\varphi dx + \beta\int_{\mathbb{R}^2} (w_{1n}^2 - w_{2n}^2) w_{1n}\varphi dx\Big| \leq \frac{C\epsilon_n}{n}.\end{aligned} \tag{3.40}$$

Using (3.34) and (3.35), we thus deduce from (3.40) that $w_{1n} \stackrel{n}{\rightharpoonup} w_1$ in $H^1(\mathbb{R}^2)$, where w_1 is a non-zero solution of

$$-\Delta w + \lambda^2 w - a^* w^3 = 0 \ \text{ in } \mathbb{R}^2, \tag{3.41}$$

and $\lambda^2 := -\lim_{n\to\infty} \mu_{1n}\epsilon_n^2 > 0$. Similar to the above argument, one can also show that $w_{2n} \stackrel{n}{\rightharpoonup} w_2$ in $H^1(\mathbb{R}^2)$, where w_2 is also a non-zero solution of (3.41). Further, we derive from (3.35) that $w_1(x) \equiv \pm w_2(x)$ a.e. in $\mathbb{R}^2$.

We next show that

$$\|w_i\|_2^2 = 1, \ \text{ where } i = 1,\, 2. \tag{3.42}$$

Indeed, since $\|w_{in}\|_2^2 \equiv 1$ and $w_i \not\equiv 0$, we have $0 < \|w_i\|_2^2 \leq 1$. On the other hand, employing (3.41) and the Pohozaev identity ([8, Lemma 8.1.2]), we derive that

$$\int_{\mathbb{R}^2} |w_i|^2 dx = \frac{1}{\lambda^2}\int_{\mathbb{R}^2} |\nabla w_i|^2 dx = \frac{a^*}{2\lambda^2}\int_{\mathbb{R}^2} |w_i|^4 dx, \ \text{ where } i = 1,\, 2.$$

We then use the Gagliardo-Nirenberg inequality (1.10) to deduce that

$$\frac{a^*}{2} \leq \frac{\int_{\mathbb{R}^2} |w_i|^2 dx \int_{\mathbb{R}^2} |\nabla w_i|^2 dx}{\int_{\mathbb{R}^2} |w_i|^4 dx} = \frac{a^*}{2}\int_{\mathbb{R}^2} |w_i|^2 dx, \tag{3.43}$$

which yields that $\|w_i\|_2^2 \geq 1$ for $i = 1$ and 2, and therefore (3.42) follows.

We now obtain from (3.42) that

$$w_{in} \stackrel{n}{\to} w_i \ \text{ strongly in } \ L^2(\mathbb{R}^2), \ \ i = 1,\, 2. \tag{3.44}$$

Since $\{\epsilon_n y_{\epsilon_n}\}$ is bounded uniformly in $\mathbb{R}^2$, there exists a subsequence, still denoted by $\{\epsilon_n\}$, of $\{\epsilon_n\}$ such that $\epsilon_n y_{\epsilon_n} \stackrel{n}{\to} z_0 \in \mathbb{R}^2$. Using Fatou's lemma, we thus obtain from (3.36) and (3.44) that

$$\begin{aligned}\hat{e}(a^* - \beta, a^* - \beta, \beta) &\geq \sum_{i=1}^{2}\int_{\mathbb{R}^2} \lim_{n\to\infty} V_i(\epsilon_n x + \epsilon_n y_{\epsilon_n})|w_{in}|^2 dx \\ &= \sum_{i=1}^{2}\int_{\mathbb{R}^2} V_i(z_0)|w_i|^2 dx = V_1(z_0) + V_2(z_0),\end{aligned}$$

which however contradicts the assumption that $\hat{e}(a^* - \beta, a^* - \beta, \beta) < \inf_{x\in\mathbb{R}^2}\big(V_1(x) + V_2(x)\big)$, and the proof of Theorem 1.3(ii) is therefore done. □

4 Uniqueness of nonnegative minimizers

The aim of this section is to prove Theorem 1.4 on the uniqueness of non-negative minimizers for (1.2) by applying the implicit function theorem. Based on the contracting map, a similar result for single GP energy functionals was also proved in [2, 28], however this methods seems not applicable to our problem.

Let λ_{i1} be the first eigenvalue of $-\Delta + V_i(x)$ in $\mathcal{H}_i$, *i.e*,

$$\lambda_{i1} = \inf\Big\{\int_{\mathbb{R}^2}\big(|\nabla u|^2 + V_i(x)\big)u^2 dx : \ u \in \mathcal{H}_i \ \text{ and } \ \int_{\mathbb{R}^2} |u|^2 dx = 1\Big\}, \ \ i = 1,\, 2. \tag{4.1}$$

Applying Lemma 2.1 (see also [32]), one can deduce that λ_{i1} is simple and can be attained by a positive normalized function $\phi_{i1} \in \mathcal{H}_i$, which is called the first eigenfunction of $-\Delta + V_i(x)$ in $\mathcal{H}_i$, $i = 1, 2$. Define

$$Z_i = \text{span}\{\phi_{i1}\}^{\perp} = \Big\{u \in \mathcal{H}_i : \int_{\mathbb{R}^2} u\phi_{i1}dx = 0\Big\}, \ \ i = 1, 2,$$

so that

$$\mathcal{H}_i = \text{span}\{\phi_{i1}\} \oplus Z_i, \ \ i = 1, 2. \tag{4.2}$$

We now recall the following properties.

Proposition 4.1. *[16, Lemma 4.2] If $V_i(x)$ satisfies (1.6) for $i = 1, 2$, then*

(i) *$ker\big(-\Delta + V_i(x) - \lambda_{i1}\big) = span\{\phi_{i1}\}$,*

(ii) *$\phi_{i1} \notin \big(-\Delta + V_i(x) - \lambda_{i1}\big)Z_i$,*

(iii) *$Im\ \big(-\Delta + V_i(x) - \lambda_{i1}\big) = \big(-\Delta + V_i(x) - \lambda_{i1}\big)Z_i$ is closed in $\mathcal{H}_i^*$,*

(iv) *$codim\ Im\ \big(-\Delta + V_i(x) - \lambda_{i1}\big) = 1$,*

where $\mathcal{H}_i^*$ denotes the dual space of $\mathcal{H}_i$ for $i = 1, 2$.

Define now the C^1 functional $F_i : \mathcal{X} \times \mathbb{R}^3 \mapsto \mathcal{H}_i^*$ by

$$F_i(u_1, u_2, \mu_i, b_i, \beta) = \big(-\Delta + V_i(x) - \mu_i\big)u_i - b_i u_i^3 - \beta u_j^2 u_i, \ \ i = 1, 2, \tag{4.3}$$

where $j \neq i$ and $j = 1, 2$. We then have the following lemma.

Lemma 4.1. *Let F_i be defined by (4.3), where $i = 1, 2$. Then there exist $\delta > 0$ and a unique function $(u_i(b_1, b_2, \beta), \mu_i(b_1, b_2, \beta)) \in C^1\big(B_\delta(\vec{0}); B_\delta(\phi_{i1}, \lambda_{i1})\big)$, where $i = 1, 2$, such that*

$$\begin{cases} \mu_i(\vec{0}) = \lambda_{i1}, \ \ u_i(\vec{0}) = \phi_{i1}, \ \ i = 1, 2; \\ F_i\big(u_1(b_1, b_2, \beta), u_2(b_1, b_2, \beta), \mu_i(b_1, b_2, \beta), b_i, \beta\big) = 0, \ \ i = 1, 2; \\ \|u_1(b_1, b_2, \beta)\|_2^2 = \|u_2(b_1, b_2, \beta)\|_2^2 = 1. \end{cases} \tag{4.4}$$

Proof. For $i = 1, 2$, define $g_i : (Z_1 \times \mathbb{R}) \times (Z_2 \times \mathbb{R}) \times \mathbb{R}^4 \mapsto \mathcal{H}_i^*$ by

$$g_i\big((z_1, \mu_1), (z_2, \mu_2), s_1, s_2, b_i, \beta\big) := F_i\big((1 + s_1)\phi_{11} + z_1, (1 + s_2)\phi_{21} + z_2, \mu_i, b_i, \beta\big).$$

Then $g_i \in C^1((Z_1 \times \mathbb{R}) \times (Z_2 \times \mathbb{R}) \times \mathbb{R}^4; \mathcal{H}_i^*)$ and

$$\begin{aligned} g_i((0, \lambda_{11}), (0, \lambda_{21}), \vec{0}) &= F_i(\phi_{11}, \phi_{21}, \lambda_{i1}, \vec{0}) = 0, \\ D_{s_i} g_i((0, \lambda_{11}), (0, \lambda_{21}), \vec{0}) &= D_{u_i} F_i(\phi_{11}, \phi_{21}, \lambda_{i1}, \vec{0})\phi_{i1} \\ &= \big(-\Delta + V_i(x) - \lambda_{i1}\big)\phi_{i1} = 0, \\ D_{s_j} g_i((0, \lambda_{11}), (0, \lambda_{21}), \vec{0}) &= D_{u_j} F_i(\phi_{11}, \phi_{21}, \lambda_{i1}, \vec{0})\phi_{j1} = 0, \end{aligned} \tag{4.5}$$

where $j \neq i, i, j = 1,\ 2$. Moreover, for any $(\hat{z}_i, \hat{\mu}_i) \in Z_i \times \mathbb{R}$, we have

$$\begin{aligned}
&\big\langle D_{(z_i,\mu_i)} g_i((0,\lambda_{11}),(0,\lambda_{21}),\vec{0}),(\hat{z}_i,\hat{\mu}_i)\big\rangle \\
=&D_{u_i}F_i(\phi_{11},\phi_{21},\lambda_{i1},\vec{0})\hat{z}_i + D_{\mu_i}F_i(\phi_{11},\phi_{21},\lambda_{i1},\vec{0})\hat{\mu}_i \\
=&\big(-\Delta + V(x) - \lambda_{i1}\big)\hat{z}_i - \hat{\mu}_i\phi_{i1} \in \mathcal{H}_i^*,
\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}
&\big\langle D_{(z_j,\mu_j)} g_i((0,\lambda_{11}),(0,\lambda_{21}),\vec{0}),(\hat{z}_j,\hat{\mu}_j)\big\rangle \\
=&D_{u_j}F_i(\phi_{11},\phi_{21},\lambda_{i1},\vec{0})\hat{z}_j + D_{\mu_j}F_i(\phi_{11},\phi_{21},\lambda_{i1},\vec{0})\hat{\mu}_j = 0.
\end{aligned} \tag{4.7}$$

It then follows from (4.6) and Proposition 4.1 that for $i = 1, 2$,

$$D_{(z_i,\mu_i)} g_i((0,\lambda_{11}),(0,\lambda_{21}),\vec{0}) : Z_i \times \mathbb{R} \mapsto \mathcal{H}_i^* \text{ is an isomorphism.} \tag{4.8}$$

We next define $G : (Z_1 \times \mathbb{R}) \times (Z_2 \times \mathbb{R}) \times \mathbb{R}^5 \mapsto \mathcal{H}_1^* \times \mathcal{H}_2^* \times \mathbb{R}^2$ by

$$G\big((z_1,\mu_1),(z_2,\mu_2),s_1,s_2,(b_1,b_2,\beta)\big) := \begin{pmatrix} g_1\big((z_1,\mu_1),(z_2,\mu_2),s_1,s_2,b_1,\beta\big) \\ g_2\big((z_1,\mu_1),(z_2,\mu_2),s_1,s_2,b_2,\beta\big) \\ \|(1+s_1)\phi_{11} + z_1\|_2^2 - 1 \\ \|(1+s_2)\phi_{21} + z_2\|_2^2 - 1 \end{pmatrix}.$$

Setting $h_i(z_i, s_i) = \|(1+s_i)\phi_{i1} + z_i\|_2^2 - 1$, we then have

$$\begin{aligned}
&D_{((z_1,\mu_1),(z_2,\mu_2),s_1,s_2)} G\big((0,\lambda_{11}),(0,\lambda_{21}),0,0,\vec{0}\,\big) \\
=&\begin{pmatrix} D_{(z_1,\mu_1)}g_1 & D_{(z_2,\mu_2)}g_1 & D_{s_1}g_1 & D_{s_2}g_1 \\ D_{(z_1,\mu_1)}g_2 & D_{(z_2,\mu_2)}g_2 & D_{s_1}g_2 & D_{s_2}g_2 \\ D_{(z_1,\mu_1)}h_1 & D_{(z_2,\mu_2)}h_1 & D_{s_1}h_1 & D_{s_2}h_1 \\ D_{(z_1,\mu_1)}h_2 & D_{(z_2,\mu_2)}h_2 & D_{s_1}h_2 & D_{s_2}h_2 \end{pmatrix} \\
=&\begin{pmatrix} (-\Delta + V_1(x) - \lambda_{11}, -\phi_{11}) & 0 & 0 & 0 \\ 0 & (-\Delta + V_2(x) - \lambda_{21}, -\phi_{21}) & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}.
\end{aligned} \tag{4.9}$$

We then derive from (4.8) that

$$D_{((z_1,\mu_1),(z_2,\mu_2),s_1,s_2)} G\big((0,\lambda_{11}),(0,\lambda_{21}),0,0,\vec{0}\big) : (Z_1 \times \mathbb{R}) \times (Z_2 \times \mathbb{R}) \times \mathbb{R}^5 \mapsto \mathcal{H}_1^* \times \mathcal{H}_2^* \times \mathbb{R}^2$$

is an isomorphism. Therefore, it follows from the implicit function theorem that there exist $\delta > 0$ and a unique function $(z_i(b_1,b_2,\beta), \mu_i(b_1,b_2,\beta), s_i(b_1,b_2,\beta)) \in C^1(B_\delta(\vec{0}); B_\delta(0,\lambda_{i1},0))$, where $i = 1,\ 2$, such that

$$\begin{cases} G\big((z_1,\mu_1),(z_2,\mu_2),s_1,s_2,(b_1,b_2,\beta)\big) = G\big((0,\lambda_{11}),(0,\lambda_{21}),0,0,\vec{0}\big) = \vec{0}, \\ z_i(\vec{0}) = 0, \;\; \mu_i(\vec{0}) = \lambda_{i1}, \;\; s_i(\vec{0}) = 0, \;\; i = 1,\ 2. \end{cases} \tag{4.10}$$

By setting

$$u_i(b_1,b_2,\beta) = (1 + s_i(b_1,b_2,\beta))\phi_{i1} + z_i(b_1,b_2,\beta)\,,\;\; (b_1,b_2,\beta) \in B_\delta(\vec{0}),\;\; i = 1,\ 2,$$

we then obtain from (4.10) that there exists a unique function $(u_i(b_1, b_2, \beta), \mu_i(b_1, b_2, \beta)) \in C^1(B_\delta(\vec{0}); B_\delta(\phi_{i1}, \lambda_{i1}))$, where $i = 1, 2$, such that

$$u_i(\vec{0}) = (1 + s_i(\vec{0}))\phi_{i1} + z_i(\vec{0}) = \phi_{i1}, \;\; \mu_i(\vec{0}) = \lambda_{i1}, \tag{4.11}$$

and

$$\begin{pmatrix} F_1\big(u_1(b_1, b_2, \beta), u_2(b_1, b_2, \beta), \mu_1(b_1, b_2, \beta), b_1, \beta\big) \\ F_2\big(u_1(b_1, b_2, \beta), u_2(b_1, b_2, \beta), \mu_2(b_1, b_2, \beta), b_2, \beta\big) \\ \|u_1(b_1, b_2, \beta)\|_2^2 - 1 \\ \|u_1(b_1, b_2, \beta)\|_2^2 - 1 \end{pmatrix} = \vec{0}, \tag{4.12}$$

and therefore (4.4) holds. This completes the proof of Lemma 4.1. □

In the following we use Lemma 4.1 to derive the uniqueness of nonnegative minimizers for sufficiently small $|(b_1, b_2, \beta)|$.

Proof of Theorem 1.4. It follows from Theorem 1.1 that $\hat{e}(b_1, b_2, \beta)$ admits at least one minimizer if $0 < b_1 < a^*$, $0 < b_2 < a^*$ and $\beta < \sqrt{(a^* - b_1)(a^* - b_2)}$. We first claim that $\hat{e}(\cdot)$ is a continuous function of (b_1, b_2, β) on the interval $I := (-\frac{a^*}{2}, \frac{a^*}{2}) \times (-\frac{a^*}{2}, \frac{a^*}{2}) \times (-\frac{a^*}{4}, \frac{a^*}{4})$. Indeed, for any $(b_1, b_2, \beta) \in I$, let (u_1, u_2) be any nonnegative minimizer of $\hat{e}(b_1, b_2, \beta)$. It then follows from (1.10) and Cauchy's inequality that

$$\hat{e}(b_1, b_2, \beta) = E_{b_1, b_2, \beta}(u_1, u_2) \geq \sum_{i=1}^{2} \Big(\frac{a^* - |b_i|}{2} - \frac{|\beta|}{2}\Big) \int_{\mathbb{R}^2} |u_i|^4 dx \geq \frac{a^*}{8} \sum_{i=1}^{2} \int_{\mathbb{R}^2} |u_i|^4 dx,$$

which implies that

$$\text{the } L^4\text{-norm of minimizers for } \hat{e}(\cdot) \text{ is bounded uniformly on the interval } I. \tag{4.13}$$

For any $(b_{i1}, b_{i2}, \beta_i) \in I$, we now denote (u_{i1}, u_{i2}) to be the corresponding nonnegative minimizer of $\hat{e}(b_{i1}, b_{i2}, \beta_i)$, where $i = 1,\, 2$. Then

$$\begin{aligned} \hat{e}(b_{11}, b_{12}, \beta_1) &= E_{b_{11}, b_{12}, \beta_1}(u_{11}, u_{12}) \\ &= E_{b_{21}, b_{22}, \beta_2}(u_{11}, u_{12}) + \frac{b_{21} - b_{11}}{2} \int_{\mathbb{R}^2} |u_{11}|^4 dx \\ &\quad + \frac{b_{22} - b_{12}}{2} \int_{\mathbb{R}^2} |u_{12}|^4 dx + (\beta_2 - \beta_1) \int_{\mathbb{R}^2} |u_{11}|^2 |u_{12}|^2 dx \\ &\geq \hat{e}(b_{21}, b_{22}, \beta_2) + O(|(b_{21}, b_{22}, \beta_2) - (b_{11}, b_{12}, \beta_1)|). \end{aligned} \tag{4.14}$$

Similarly, we also have

$$\hat{e}(b_{21}, b_{22}, \beta_2) \geq \hat{e}(b_{11}, b_{12}, \beta_1) + O(|(b_{21}, b_{22}, \beta_2) - (b_{11}, b_{12}, \beta_1)|). \tag{4.15}$$

We then derive from (4.14) and (4.15) that

$$\lim_{(b_{11}, b_{12}, \beta_1) \to (b_{21}, b_{22}, \beta_2)} \hat{e}(b_{11}, b_{12}, \beta_1) = \hat{e}(b_{21}, b_{22}, \beta_2),$$

which implies that $\hat{e}(\cdot)$ is continuous on the interval I, and the above claim is therefore proved.

Let $(u_{b_1,\beta}, u_{b_2,\beta})$ be a non-negative minimizer of $\hat{e}(b_1, b_2, \beta)$ with $(b_1, b_2, \beta) \in I$. We then deduce that $(u_{b_1,\beta}, u_{b_2,\beta})$ satisfies the Euler-Lagrange system

$$\begin{cases} (-\Delta + V_1(x) - \mu_{b_1,\beta})u_{b_1,\beta} - b_1 u^3_{b_1,\beta} - \beta u^2_{b_2,\beta} u_{b_1,\beta} = 0 \ \text{ in } \mathbb{R}^2, \\ (-\Delta + V_2(x) - \mu_{b_2,\beta})u_{b_2,\beta} - b_2 u^3_{b_2,\beta} - \beta u^2_{b_1,\beta} u_{b_2,\beta} = 0 \ \text{ in } \mathbb{R}^2, \end{cases} \tag{4.16}$$

i.e,

$$F_i(u_{b_1,\beta}, u_{b_2,\beta}, \mu_{b_i,\beta}, b_i, \beta) = 0, \ \ i = 1, 2, \tag{4.17}$$

where $(\mu_{b_1,\beta}, \mu_{b_2,\beta}) \in \mathbb{R}^2$ is a Lagrange multiplier. We then derive from (4.13) and the above claim that

$$\begin{aligned} & E_{0,0,0}(u_{b_1,\beta}, u_{b_2,\beta}) \\ =& E_{b_1,b_2,\beta}(u_{b_1,\beta}, u_{b_2,\beta}) + \sum_{i=1}^{2} \frac{b_i}{2} \int_{\mathbb{R}^2} |u_{b_i,\beta}|^4 dx + \beta \int_{\mathbb{R}^2} |u_{b_1,\beta}|^2 |u_{b_2,\beta}|^2 dx \\ =& \hat{e}(b_1, b_2, \beta) + O(|(b_1, b_2, \beta)|) \to \hat{e}(0, 0, 0) \quad \text{as} \quad (b_1, b_2, \beta) \to 0. \end{aligned} \tag{4.18}$$

On the other hand, one can check easily that $\hat{e}(0, 0, 0) = \lambda_{11} + \lambda_{21}$, and (ϕ_{11}, ϕ_{21}) is the unique non-negative minimizer of $\hat{e}(0, 0, 0)$, where $(\lambda_{i1}, \phi_{i1})$ is the first eigenpair of $-\Delta + V_i(x)$ in $\mathcal{H}_i$, $i = 1, 2$. We then deduce from Lemma 2.1 that

$$u_{b_i,\beta} \to \phi_{i1} \ \text{ in } \ \mathcal{H}_i \ \text{ as } \ (b_1, b_2, \beta) \to (0, 0, 0), \ \ i = 1, 2. \tag{4.19}$$

Moreover, by (4.16) and (4.19) we have

$$\begin{aligned} \mu_{b_i,\beta} =& \int_{\mathbb{R}^2} |\nabla u_{b_i,\beta}|^2 + V_i(x)|u_{b_i,\beta}|^2 dx - b_i \int_{\mathbb{R}^2} |u_{b_i,\beta}|^4 dx - \beta \int_{\mathbb{R}^2} |u_{b_1,\beta}|^2 |u_{b_2,\beta}|^4 dx \\ & \to \lambda_{i1} \quad \text{as} \ \ (b_1, b_2, \beta) \to (0, 0, 0), \quad i = 1, 2. \end{aligned} \tag{4.20}$$

Applying (4.19) and (4.20), we then derive that there exists a small constant $\delta_1 > 0$ such that

$$\|u_{b_i,\beta} - \phi_{i1}\|_{\mathcal{H}_i} < \delta_1 \ \text{ and } \ |\mu_{b_i,\beta} - \lambda_{i1}| < \delta_1 \ \text{ if } \ (b_1, b_2, \beta) \in B_{\delta_1}(\vec{0}), \ \ i = 1, 2. \tag{4.21}$$

We thus conclude from (4.17) and Lemma 4.1 that

$$\mu_{b_i,\beta} = \mu_i(b_1, b_2, \beta), \ u_{b_i,\beta} = u_i(b_1, b_2, \beta), \ \text{ if } \ (b_1, b_2, \beta) \in B_{\delta_1}(\vec{0}), \ i = 1, 2.$$

This therefore implies that for sufficiently small $|(b_1, b_2, \beta)|$, $(u_1(b_1, b_2, \beta),\, u_2(b_1, b_2, \beta))$ is a unique non-negative minimizer of $\hat{e}(b_1, b_2, \beta)$, and we are done. □

5 Optimal Estimates

To simplify the notations and the proof, we denote $a_i = b_i + \beta > 0$ for any fixed $0 < \beta < a^*$, where $i = 1, 2$. It is then equivalent to rewriting the functional (1.5) as

$$\begin{aligned} E_{a_1,a_2}(u_1, u_2) :=& \sum_{i=1}^{2} \int_{\mathbb{R}^2} \Big(|\nabla u_i|^2 + V_i(x)|u_i|^2 - \frac{a_i}{2}|u_i|^4 \Big) dx \\ & + \frac{\beta}{2} \int_{\mathbb{R}^2} \big(|u_1|^2 - |u_2|^2 \big)^2 dx \,, \quad \text{where} \quad (u_1, u_2) \in \mathcal{X}. \end{aligned} \tag{5.1}$$

Also, the minimization problem (1.2) is then equivalent to the following one:

$$e(a_1, a_2) := \inf_{\{(u_1,u_2)\in\mathcal{M}\}} E_{a_1,a_2}(u_1, u_2), \tag{5.2}$$

where $\mathcal{M}$ is defined by (1.3).

To prove Theorem 1.5, we first need to establish the following crucial optimal energy estimates as $(a_1, a_2) \nearrow (a^*, a^*)$.

Proposition 5.1. *Suppose $0 < \beta < a^*$ and $V_i(x)$ satisfies (1.17) and (1.18), where $i = 1, 2$. Then there exist two positive constants C_1 and C_2, independent of a_1 and a_2, such that*

$$C_1\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}} \le e(a_1, a_2) \le C_2\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}} \quad as \quad (a_1, a_2) \nearrow (a^*, a^*), \tag{5.3}$$

where $p_0 > 0$ is defined by (1.20), and $e(a_1, a_2)$ is defined by (5.2). Moreover, if (u_{a_1}, u_{a_2}) is a non-negative minimizer of $e(a_1, a_2)$, then there exist two positive constants C_3 and C_4, independent of a_1 and a_2, such that

$$C_3\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{-\frac{2}{p_0+2}} \le \int_{\mathbb{R}^2} |u_{a_i}|^4 dx \le C_4\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{-\frac{2}{p_0+2}} \tag{5.4}$$

as $(a_1, a_2) \nearrow (a^*, a^*)$.

We remark that even though the upper bound of (5.3) can be proved similarly to that of Lemma 3 in [14], the arguments of [14] do not give the lower bound of (5.3). For this reason, as discussed below, we need employ a little more involved analysis to address the optimal lower bound of (5.3).

In what follows, we focus on the proof of Proposition 5.1. For any fixed $0 < \beta < a^*$, denote (u_{a_1}, u_{a_2}) to be a non-negative minimizer of (5.2). We start with the following energy estimates of $e(a_1, a_2)$.

Lemma 5.1. *Under the assumptions of Proposition 5.1, there exists a constant $C > 0$, independent of a_1 and a_2, such that*

$$e_1(a_1) + e_2(a_2) + \frac{\beta}{2}\int_{\mathbb{R}^2} \big(|u_{a_1}|^2 - |u_{a_2}|^2\big)^2 dx \le e(a_1, a_2) \le C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}} \tag{5.5}$$

as $(a_1, a_2) \nearrow (a^*, a^*)$, where $e_i(\cdot)$ is given by (1.7) for $i = 1, 2$.

Proof. Since (u_{a_1}, u_{a_2}) is a non-negative minimizer of (5.2), we note from (1.8) that

$$E_{a_1,a_2}(u, v) = E^1_{a_1}(u) + E^2_{a_2}(v) + \frac{\beta}{2}\int_{\mathbb{R}^2} \big(|u|^2 - |v|^2\big)^2 dx \quad \text{for all } (u, v) \in \mathcal{X},$$

where $E^i_{a_i}(\cdot)$ is defined by (1.8) for $i = 1$ and 2. This relation then implies that

$$e(a_1, a_2) \ge e_1(a_1) + e_2(a_2) + \frac{\beta}{2}\int_{\mathbb{R}^2} \big(|u_{a_1}|^2 - |u_{a_2}|^2\big)^2 dx,$$

which gives the lower bound of (5.5).

Stimulated by Lemma 3 in [14], we next prove the upper bound of (5.5) as follows. Without loss of generality, we may assume $p_0 = \bar{p}_1 = \min\{p_{11}, p_{21}\} > 0$ and $p_{11} \le p_{21}$, where p_0 and $\bar{p}_i$ are defined by (1.20). We proceed similarly to the proof of Lemma A2 in the Appendix, and use the trial function (ϕ, ϕ) for ϕ satisfying (A.3) with $\bar{x}_0 = x_{11}$. Choose $R > 0$ small enough that

$$V_i(x) \le C|x - x_{11}|^{p_{i1}} \quad \text{for} \quad |x - x_{11}| \le R, \quad i = 1,\, 2.$$

By the exponential decay of $Q(x)$, we have

$$\int_{\mathbb{R}^2} V_i(x)\phi^2(x)dx \le C\tau^{-p_{i1}} \int_{\mathbb{R}^2} |x|^{p_{i1}} Q^2(x)dx \le C\tau^{-p_{i1}} \quad \text{as} \quad \tau \to \infty\,, \quad i = 1,\, 2.$$

This inequality and (A.5) then imply that

$$E_{a_1,a_2}(\phi, \phi) \le \frac{2}{a^*}\Big(a^* - \frac{a_1 + a_2}{2}\Big)\tau^2 + C\big(\tau^{-p_{11}} + \tau^{-p_{21}}\big).$$

Setting $\tau = \Big(a^* - \frac{a_1+a_2}{2}\Big)^{\frac{-1}{p_0+2}}$ and using $p_0 = p_{11} \le p_{21}$, we derive that

$$e(a_1, a_2) \le E_{a_1,a_2}(\phi, \phi) \le C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}}\,,$$

which therefore gives the upper bound of (5.5). □

By Lemma 5.1, for $i = 1,\, 2$, we have

$$e_i(a_i) \le E^i_{a_i}(u_{a_i}) \le C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}} \tag{5.6}$$

as $(a_1, a_2) \nearrow (a^*, a^*)$, where $e_i(\cdot)$ is defined by (1.7). On the other hand, it is proved in [14, Lemma 3] that for

$$p_i := \max\{p_{ij}, j = 1, \ldots, n_i\} > 0\,, \quad i = 1,\, 2\,, \tag{5.7}$$

there exists two positive constants m and M, independent of a_1 and a_2, such that

$$m(a^* - a_i)^{\frac{p_i}{p_i+2}} \le e_i(a_i) \le M(a^* - a_i)^{\frac{p_i}{p_i+2}} \ \text{ for } \ 0 \le a_i \le a^*\,, \ \text{and} \ \ i = 1,\, 2. \tag{5.8}$$

Applying (5.8) and Lemma 5.1, we next derive the following $L^4(\mathbb{R}^2)-$estimates of minimizers.

Lemma 5.2. *Under the assumptions of Proposition 5.1, we have*

$$C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{-\frac{2}{p_0+2}\frac{p_0}{p_i}} \le \int_{\mathbb{R}^2} |u_{a_i}|^4 dx \le \frac{1}{C}\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{-\frac{2}{p_0+2}} \ \textit{as } (a_1, a_2) \nearrow (a^*, a^*), \tag{5.9}$$

and

$$\lim_{(a_1,a_2)\nearrow(a^*,a^*)} \frac{\int_{\mathbb{R}^2} |u_{a_1}|^4 dx}{\int_{\mathbb{R}^2} |u_{a_2}|^4 dx} = 1, \tag{5.10}$$

where $p_i \ge p_0$, and $p_i > 0$ is given by (5.7) for $i = 1$ and 2.

Proof. We first prove the lower bound of (5.9). Pick any $0 < b < a_i < a^*$ ($i = 1, 2$), and observe that

$$e_i(b) \leq E^i_{a_i}(u_{a_i}) + \frac{a_i - b}{2} \int_{\mathbb{R}^2} |u_{a_i}|^4 dx, \quad i = 1, 2.$$

It then follows from (5.6) and (5.8) that

$$\begin{aligned} \frac{1}{2} \int_{\mathbb{R}^2} |u_{a_i}|^4 dx &\geq \frac{e_i(b) - C\big(a^* - \frac{a_1+a_2}{2}\big)^{\frac{p_0}{p_0+2}}}{a_i - b} \\ &\geq \frac{m(a^* - b)^{\frac{p_i}{p_i+2}} - C\big(a^* - \frac{a_1+a_2}{2}\big)^{\frac{p_0}{p_0+2}}}{a_i - b}. \end{aligned} \tag{5.11}$$

Take $b = a_i - C_1\big(a^* - \frac{a_1+a_2}{2}\big)^{\frac{p_0(p_i+2)}{p_i(p_0+2)}}$, where $C_1 > 0$ is so large that $mC_1^{\frac{p_0(p_i+2)}{p_i(p_0+2)}} > 2C$. We then derive from (5.11) that

$$\int_{\mathbb{R}^2} |u_{a_i}|^4 dx \geq C_2\big(a^* - \frac{a_1 + a_2}{2}\big)^{-\frac{2}{p_0+2}\frac{p_0}{p_i}}, \quad i = 1, 2, \tag{5.12}$$

which therefore implies the lower bound of (5.9).

We next prove the upper bound of (5.9). One can note from (5.6) that for $i = 1, 2$,

$$E_{a_i}(u_{a_i}) \leq C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}} \quad \text{as } (a_1, a_2) \nearrow (a^*, a^*). \tag{5.13}$$

Without loss of generality, we may assume that $a_1 \leq a_2 \leq a^*$ and $(a_1, a_2) \neq (a^*, a^*)$. By (1.10), we then have

$$E^1_{a_1}(u_{a_1}) \geq \frac{a^* - a_1}{2} \int_{\mathbb{R}^2} |u_{a_1}|^4 dx \geq \frac{1}{2}\Big(a^* - \frac{a_1 + a_2}{2}\Big) \int_{\mathbb{R}^2} |u_{a_1}|^4 dx.$$

It thus follows from (5.13) that the upper bound of (5.9) holds for u_{a_1}. Similarly, the upper bound of (5.9) holds also for u_{a_2} if (5.10) is true, and then the proof is done.

Now we come to prove (5.10). Recall from Lemma 5.1 that

$$\int_{\mathbb{R}^2} \big(|u_{a_1}|^2 - |u_{a_2}|^2\big)^2 dx \leq C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{p_0+2}} \quad \text{as} \quad (a_1, a_2) \nearrow (a^*, a^*),$$

which implies that

$$\begin{aligned} &\Big| \int_{\mathbb{R}^2} |u_{a_1}|^4 dx - \int_{\mathbb{R}^2} |u_{a_2}|^4 dx \Big| \\ &\leq \Big(\int_{\mathbb{R}^2} \big(|u_{a_1}|^2 - |u_{a_2}|^2\big)^2 dx \Big)^{\frac{1}{2}} \Big(\int_{\mathbb{R}^2} \big(|u_{a_1}|^2 + |u_{a_2}|^2\big)^2 dx \Big)^{\frac{1}{2}} \\ &\leq C\Big(a^* - \frac{a_1 + a_2}{2}\Big)^{\frac{p_0}{2(p_0+2)}} \Big[\Big(\int_{\mathbb{R}^2} |u_{a_1}|^4 dx \Big)^{\frac{1}{2}} + \Big(\int_{\mathbb{R}^2} |u_{a_2}|^4 dx \Big)^{\frac{1}{2}} \Big] \end{aligned} \tag{5.14}$$

as $(a_1, a_2) \nearrow (a^*, a^*)$. Since it follows from (5.12) that $\int_{\mathbb{R}^2} |u_{a_i}|^4 dx \to \infty$ as $(a_1, a_2) \nearrow (a^*, a^*)$, where $i = 1, 2$, we conclude (5.10) from the above estimate. □

We next claim that the upper estimates of (5.5) and (5.9) are optimal. By Lemma 5.1, we see that

$$\sum_{i=1}^{2}\int_{\mathbb{R}^2} V_i(x)|u_{a_i}(x)|^2dx \le e(a_1,a_2) \le C\big(a^* - \frac{a_1+a_2}{2}\big)^{\frac{p_0}{p_0+2}} \quad \text{as} \quad (a_1,a_2)\nearrow (a^*,a^*). \tag{5.15}$$

Set

$$\epsilon^{-2}(a_1,a_2) := \int_{\mathbb{R}^2} |u_{a_1}(x)|^4dx, \quad \text{where } \ \epsilon(a_1,a_2)>0\,. \tag{5.16}$$

It then yields from (5.9) that $\epsilon(a_1,a_2)\searrow 0$ as $(a_1,a_2)\nearrow(a^*,a^*)$. Moreover, we deduce from (5.5) and (5.10) that for $i=1,\,2$,

$$\int_{\mathbb{R}^2}|u_{a_i}|^4dx,\int_{\mathbb{R}^2}|\nabla u_{a_i}|^2dx \sim \epsilon^{-2}(a_1,a_2) \quad \text{as} \quad (a_1,a_2)\nearrow(a^*,a^*), \tag{5.17}$$

where $f\sim g$ means that f/g is bounded from below and above. In view of above facts, we next define the $L^2(\mathbb{R}^2)$-normalized function

$$\tilde{w}_{a_i}(x) := \epsilon(a_1,a_2)u_{a_i}\big(\epsilon(a_1,a_2)x\big), \ \ i=1,\,2. \tag{5.18}$$

It then follows from (5.10), (5.16) and (5.17) that for $i=1,\,2$,

$$\int_{\mathbb{R}^2}|\tilde{w}_{a_i}|^4dx = 1 \quad \text{and} \quad m \le \int_{\mathbb{R}^2}|\nabla\tilde{w}_{a_i}|^2dx \le \frac{1}{m} \quad \text{as} \quad (a_1,a_2)\nearrow(a^*,a^*), \tag{5.19}$$

where $m>0$ is independent of a_1 and a_2. In the rest part of this section, for convenience we use $\epsilon>0$ to denote $\epsilon(a_1,a_2)$ so that $\epsilon\searrow 0$ as $(a_1,a_2)\nearrow(a^*,a^*)$.

Lemma 5.3. *Under the assumptions of Proposition 5.1, we have*

(i). *There exist a sequence $\{y_\epsilon\}$ as well as positive constants R_0 and η_i such that for $i=1,\,2$, the function*

$$w_{a_i}(x) = \tilde{w}_{a_i}(x+y_\epsilon) = \epsilon u_{a_i}(\epsilon x+\epsilon y_\epsilon) \tag{5.20}$$

satisfies

$$\liminf_{\epsilon\to 0}\int_{B_{R_0}(0)}|w_{a_i}|^2dx \ge \eta_i > 0. \tag{5.21}$$

(ii). *The following estimate holds*

$$\lim_{(a_1,a_2)\nearrow(a^*,a^*)} \operatorname{dist}(\epsilon y_\epsilon,\Lambda) = 0, \tag{5.22}$$

where $\Lambda = \big\{x\in\mathbb{R}^2:\ V_1(x)=V_2(x)=0\big\}$ is given by (1.19). Moreover, for any sequence $\{(a_{1k},a_{2k})\}$ satisfying $(a_{1k},a_{2k})\nearrow(a^*,a^*)$ as $k\to\infty$, there exists a convergent subsequence of $\{(a_{1k},a_{2k})\}$, still denoted by $\{(a_{1k},a_{2k})\}$, such that

$$\epsilon_k y_{\epsilon_k} \overset{k}{\to} x_0\in\Lambda \ \textit{and} \ w_{a_{ik}} \overset{k}{\to} w_0 \quad \textit{strongly in } H^1(\mathbb{R}^2), \quad i=1,\ 2, \tag{5.23}$$

where w_0 satisfies

$$w_0(x) = \frac{\lambda}{\|Q\|_2}Q(\lambda|x-y_0|) \quad \textit{for some } y_0\in\mathbb{R}^2 \textit{ and } \lambda>0. \tag{5.24}$$

Proof. (i). In order to establish (5.21), in view of (5.20) it suffices to prove that there exist $R_0 > 0$ and $\eta_i > 0$ such that

$$\liminf_{\epsilon \to 0} \int_{B_{R_0}(y_\epsilon)} |\tilde{w}_{a_i}|^2 dx \geq \eta_i > 0\,, \quad \text{where} \quad i = 1,\, 2. \tag{5.25}$$

We first show that (5.25) holds for $\tilde{w}_{a_1}$. Indeed, if it is false, then for any $R > 0$, there exists a subsequence $\{\tilde{w}_{a_{1k}}\}$, where $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$ as $k \to \infty$, such that

$$\lim_{k \to \infty} \sup_{y \in \mathbb{R}^2} \int_{B_R(y)} |\tilde{w}_{a_{1k}}|^2 dx = 0.$$

By Lemma I.1 in [26], we then deduce that $\tilde{w}_{a_{1k}} \xrightarrow{k} 0$ in $L^p(\mathbb{R}^2)$ for any $2 < p < \infty$, which contradicts (5.19). Thus $\tilde{w}_{a_1}$ satisfies (5.25) for a sequence $\{y_\epsilon\}$, $R_0 > 0$ and $\eta_1 > 0$.

We next show that for the sequence $\{y_\epsilon\}$, $R_0 > 0$ and $\eta_1 > 0$ obtained above, (5.25) holds also for $\tilde{w}_{a_2}$ with some constant $\eta_2 > 0$. On the contrary, if (5.25) is false for $\tilde{w}_{a_2}$, then there exists a subsequence $\{\tilde{w}_{a_{2k}}\}$, where $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$ as $k \to \infty$, such that

$$\lim_{k \to \infty} \sup \int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{2k}}|^2 dx = 0,$$

where $\epsilon_k := \epsilon(a_{1k}, a_{2k}) > 0$ is defined by (5.16). Since $\tilde{w}_{a_i}$ is bounded uniformly in $H^1(\mathbb{R}^2) \cap L^\gamma(\mathbb{R}^2)$ for all $2 \leq \gamma < \infty$, we may choose $\gamma > 4$ and $\theta \in (0, 1)$ such that $\frac{1}{4} = \frac{1-\theta}{\gamma} + \frac{\theta}{2}$. It then follows from the Hölder inequality that

$$\begin{aligned}
\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{1k}}|^2 |\tilde{w}_{a_{2k}}|^2 dx &\leq \Big(\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{1k}}|^4 dx \Big)^{\frac{1}{2}} \Big(\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{2k}}|^4 dx \Big)^{\frac{1}{2}} \\
&\leq C \Big(\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{2k}}|^\gamma dx \Big)^{\frac{2(1-\theta)}{\gamma}} \Big(\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{2k}}|^2 dx \Big)^{\theta} \\
&\leq C \Big(\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{2k}}|^2 dx \Big)^{\theta} \to 0 \quad \text{as} \quad k \to \infty.
\end{aligned}$$

By applying the estimate (5.25) for $\tilde{w}_{a_{1k}}$, we use again the Hölder inequality to derive from the above that, for k large,

$$\begin{aligned}
\int_{B_{R_0}(y_{\epsilon_k})} (|\tilde{w}_{a_{1k}}|^2 - |\tilde{w}_{a_{2k}}|^2)^2 dx &\geq \frac{1}{2} \int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{1k}}|^4 dx \\
&\geq \frac{1}{2\pi R_0^2} \Big(\int_{B_{R_0}(y_{\epsilon_k})} |\tilde{w}_{a_{1k}}|^2 dx \Big)^2 \geq \frac{\eta_1^2}{2\pi R_0^2},
\end{aligned}$$

a contradiction, since Lemma 5.1 implies that

$$\int_{\mathbb{R}^2} \big(|\tilde{w}^2_{a_{1k}} - |\tilde{w}_{a_{2k}}|^2\big)^2 dx = \epsilon_k^2 \int_{\mathbb{R}^2} \big(|u_{a_{1k}}|^2 - |u_{a_{2k}}|^2\big)^2 dx \to 0 \quad \text{as} \quad k \to \infty.$$

Therefore, (5.25) holds also for $\tilde{w}_{a_2}$ with some constant $\eta_2 > 0$, and part (i) is proved.

(ii). We first prove (5.22). By (5.15) we note that

$$\sum_{i=1}^{2}\int_{\mathbb{R}^2} V_i(x)|u_{a_i}(x)|^2dx = \sum_{i=1}^{2}\int_{\mathbb{R}^2} V_i(\epsilon x+\epsilon y_\epsilon)|w_{a_i}(x)|^2dx \to 0 \tag{5.26}$$

as $(a_1, a_2) \nearrow (a^*, a^*)$. On the contrary, suppose (5.22) is incorrect, then there exist $\delta > 0$ and a subsequence $\{(a_{1n}, a_{2n})\}$, which satisfies $(a_{1n}, a_{2n}) \nearrow (a^*, a^*)$ as $n \to \infty$, such that

$$\epsilon_n := \epsilon(a_{1n}, a_{2n}) \to 0 \quad \text{and} \quad \mathrm{dist}(\epsilon_n y_{\epsilon_n}, \Lambda) \geq \delta > 0 \quad \text{as} \quad n \to \infty.$$

This implies that there exists $C = C(\delta) > 0$ such that

$$\lim_{n\to\infty} V_1(\epsilon_n y_{\epsilon_n}) + V_2(\epsilon_n y_{\epsilon_n}) \geq C(\delta) > 0.$$

Hence, by Fatou's Lemma and (5.21) we obtain that

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{i=1}^{2}\int_{\mathbb{R}^2} V_i(\epsilon_n x+\epsilon_n y_{\epsilon_n})|w_{a_{in}}(x)|^2dx\\
&\geq \sum_{i=1}^{2}\int_{B_{R_0}(0)} \liminf_{n\to\infty} V_i(\epsilon_n x+\epsilon_n y_{\epsilon_n})|w_{a_{in}}(x)|^2dx \geq C(\delta)\min\{\eta_1,\eta_2\},
\end{aligned}$$

which however contradicts (5.26). Therefore, (5.22) holds.

We prove now (5.23) and (5.24). Since (u_{a_1}, u_{a_2}) is a non-negative minimizer of (5.2), it satisfies the Euler-Lagrange system

$$\begin{cases} -\Delta u_{a_1} + V_1(x)u_{a_1} = \mu_{a_1}u_{a_1} + a_1u_{a_1}^3 - \beta(u_{a_1}^2 - u_{a_2}^2)u_{a_1} & \text{in} \quad \mathbb{R}^2\,, \\ -\Delta u_{a_2} + V_2(x)u_{a_2} = \mu_{a_2}u_{a_2} + a_2u_{a_2}^3 - \beta(u_{a_2}^2 - u_{a_1}^2)u_{a_2} & \text{in} \quad \mathbb{R}^2, \end{cases} \tag{5.27}$$

where (μ_{a_1}, μ_{a_2}) is a suitable Lagrange multiplier, and

$$\mu_{a_i} = E^i_{a_i}(u_{a_i}) - \frac{a_i}{2}\int_{\mathbb{R}^2}|u_{a_i}|^4dx - \beta\int_{\mathbb{R}^2}(-1)^i(u_{a_1}^2 - u_{a_2}^2)u_{a_i}^2dx, \quad i = 1,\, 2\,.$$

It then follows from Lemma 5.1, (5.1), (5.10) and (5.17) that for $i = 1,\, 2$,

$$\mu_{a_i} \sim -\frac{a_i}{2}\int_{\mathbb{R}^2}|u_{a_i}|^4dx \sim -\epsilon^{-2} \quad \text{and} \quad \mu_{a_1}/\mu_{a_2} \to 1 \quad \text{as} \quad (a_1, a_2) \nearrow (a^*, a^*). \tag{5.28}$$

Note also from (5.20) that $w_{a_i}(x)$ defined in (5.20) satisfies the elliptic system

$$\begin{cases} -\Delta w_{a_1} + \epsilon^2V_1(\epsilon x+\epsilon y_\epsilon)w_{a_1} = \epsilon^2\mu_{a_1}w_{a_1} + a_1w_{a_1}^3 - \beta(w_{a_1}^2 - w_{a_2}^2)w_{a_1} & \text{in} \ \ \mathbb{R}^2\,, \\ -\Delta w_{a_2} + \epsilon^2V_2(\epsilon x+\epsilon y_\epsilon)w_{a_2} = \epsilon^2\mu_{a_2}w_{a_2} + a_2w_{a_2}^3 - \beta(w_{a_2}^2 - w_{a_1}^2)w_{a_2} & \text{in} \ \ \mathbb{R}^2, \end{cases} \tag{5.29}$$

where the Lagrange multiplier (μ_{a_1}, μ_{a_2}) satisfies (5.28).

For any given sequence $\{(a_{1k}, a_{2k})\}$ with $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$ as $k \to \infty$, we deduce from (5.22) and (5.28) that there exists a subsequence of $\{(a_{1k}, a_{2k})\}$, still denoted by $\{(a_{1k}, a_{2k})\}$, such that

$$\epsilon_k y_{\epsilon_k} \xrightarrow{k} x_0 \in \Lambda, \quad \epsilon_k^2\mu_{a_{ik}} \xrightarrow{k} -\lambda^2 < 0 \quad \text{for some} \quad \lambda > 0, \tag{5.30}$$

and

$$w_{a_{ik}} \overset{k}{\rightharpoonup} w_i \geq 0 \text{ weakly in } H^1(\mathbb{R}^2) \text{ for some } w_i \in H^1(\mathbb{R}^2),\ \ i = 1,\, 2.$$

Since (5.5) implies that

$$\|w_{a_1}^2 - w_{a_2}^2\|_2 = \epsilon \|u_{a_1}^2 - u_{a_2}^2\|_2 \to 0 \quad \text{as} \quad (a_1, a_2) \nearrow (a^*, a^*), \tag{5.31}$$

we have $w_1 = w_2 \geq 0$ a.e. in $\mathbb{R}^2$. We thus write $0 \leq w_0 := w_1 = w_2 \in H^1(\mathbb{R}^2)$. Passing to the weak limit in (5.29), we deduce from (5.30) and (5.31) that w_0 satisfies

$$-\Delta w_0(x) = -\lambda^2 w_0(x) + a^* w_0^3(x) \text{ in } \mathbb{R}^2. \tag{5.32}$$

Furthermore, it follows from (5.21) and the strong maximum principle that $w_0 > 0$. By a simple rescaling, the uniqueness (up to translations) of positive solutions for the equation (1.9) implies that

$$w_0(x) = \frac{\lambda}{\|Q\|_2} Q(\lambda |x - y_0|) \text{ for some } y_0 \in \mathbb{R}^2. \tag{5.33}$$

Note that $||w_0||_2^2 = 1$, by the norm preservation we further conclude that

$$w_{a_{ik}} \overset{k}{\to} w_0 \text{ strongly in } L^2(\mathbb{R}^2).$$

Moreover, this strong convergence holds also for all $p \geq 2$, since $\{w_{a_{ik}}\}$ is bounded in $H^1(\mathbb{R}^2)$. Then, note that $w_{a_{ik}}$ and w_0 satisfy (5.29) and (5.32), respectively, a simple analysis shows that

$$w_{a_{ik}} \overset{k}{\to} w_0 \text{ strongly in } H^1(\mathbb{R}^2), \quad i = 1,\, 2.$$

Therefore, (5.23) and (5.24) are established. □

Applying above lemmas, we finally prove Proposition 5.1 on the optimal estimates of $e(a_1, a_2)$.

Proof of Proposition 5.1. For any sequence $\{(a_{1k}, a_{2k})\}$ satisfying $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$ as $k \to \infty$, it follows from Lemma 5.3 (ii) that there exists a convergent subsequence, still denoted by $\{(a_{1k}, a_{2k})\}$, such that (5.23) holds and $\epsilon_k y_{\epsilon_k} \overset{k}{\to} x_0 \in \Lambda$. Without loss of generality, we may assume $x_0 = x_{1j_0}$ for some $1 \leq j_0 \leq l$. We first claim that

$$\limsup_{k\to\infty} \frac{|\epsilon_k y_{\epsilon_k} - x_{1j_0}|}{\epsilon_k} < \infty. \tag{5.34}$$

Actually, by (5.23) and (5.24), we have for some $R_0 > 0$,

$$\begin{aligned}
e(a_{1k}, a_{2k}) =& E_{a_{1k},a_{2k}}(u_{a_{1k}}, u_{a_{2k}}) \\
\geq& \sum_{i=1}^{2} \Big\{ \frac{1}{\epsilon_k^2} \Big[\int_{\mathbb{R}^2} |\nabla w_{a_{ik}}(x)|^2 dx - \frac{a^*}{2} \int_{\mathbb{R}^2} |w_{a_{ik}}(x)|^4 dx \Big] \\
&+ \frac{a^* - a_{ik}}{2\epsilon_k^2} \int_{\mathbb{R}^2} |w_{a_{ik}}(x)|^4 dx + \int_{\mathbb{R}^2} V_i(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_{a_{ik}}(x)|^2 dx \Big\} \\
\geq& \sum_{i=1}^{2} \Big\{ \frac{a^* - a_{ik}}{4\epsilon_k^2} \int_{\mathbb{R}^2} |w_0(x)|^4 dx + \int_{B_{R_0}(0)} V_i(\epsilon_k x + \epsilon_k y_{\epsilon_k}) |w_{a_{ik}}(x)|^2 dx \Big\} \\
\geq& C_1 \frac{2a^* - a_{1k} - a_{2k}}{\epsilon_k^2} + C_2 \sum_{i=1}^{2} \epsilon_k^{p_{ij_0}} \int_{B_{R_0}(0)} \Big| x + \frac{\epsilon_k y_{\epsilon_k} - x_{1j_0}}{\epsilon_k} \Big|^{p_{ij_0}} |w_{a_{ik}}(x)|^2 dx.
\end{aligned} \tag{5.35}$$

Suppose now that there exists a subsequence such that $\frac{|\epsilon_k y_{\epsilon_k} - x_{1j_0}|}{\epsilon_k} \to \infty$ as $k \to \infty$. By using Fatou's Lemma, we then deduce from (5.35) and (5.21) that for any $M > 0$,

$$\begin{aligned}
e(a_{1k}, a_{2k}) \geq& C_1 \frac{2a^* - a_{1k} - a_{2k}}{\epsilon_k^2} + C_2 M \epsilon_k^{\bar{p}_{j_0}} \geq C M^{\frac{1}{\bar{p}_{j_0}+2}} \Big(a^* - \frac{a_{1k} + a_{2k}}{2} \Big)^{\frac{\bar{p}_{j_0}}{\bar{p}_{j_0}+2}} \\
\geq& C M^{\frac{1}{\bar{p}_{j_0}+2}} \Big(a^* - \frac{a_{1k} + a_{2k}}{2} \Big)^{\frac{p_0}{p_0+2}},
\end{aligned}$$

where $p_0 = \max_{1 \leq j \leq l} \bar{p}_j$ and $\bar{p}_{j_0} = \min\{p_{1j_0}, p_{2j_0}\} > 0$ are given by (1.20). This estimate however contradicts the upper bound of (5.5), and the claim (5.34) is thus established.

We now deduce from (5.34) that there exists a subsequence of $\{\epsilon_k\}$, still denoted by $\{\epsilon_k\}$, such that

$$\frac{\epsilon_k y_{\epsilon_k} - x_{1j_0}}{\epsilon_k} \to y_1 \text{ as } k \to \infty$$

holds for some $y_1 \in \mathbb{R}^2$. By applying (5.35), then there exists a constant $C_1 > 0$, independent of a_{1k} and a_{2k}, such that

$$e(a_{1k}, a_{2k}) \geq C_1 \Big(a^* - \frac{a_{1k} + a_{2k}}{2} \Big)^{\frac{\bar{p}_{j_0}}{\bar{p}_{j_0}+2}} \text{ as } (a_{1k}, a_{2k}) \nearrow (a^*, a^*).$$

Since $\bar{p}_{j_0} \leq p_0$, applying the upper bound of (5.5), we conclude from the above estimate that $\bar{p}_{j_0} = p_0$ and (5.3) holds for the above subsequence $\{(a_{1k}, a_{2k})\}$.

We next prove that (5.4) holds for the above subsequence $\{(a_{1k}, a_{2k})\}$. Suppose that

$$\text{either} \quad \epsilon_k >> \Big(a^* - \frac{a_{1k} + a_{2k}}{2} \Big)^{\frac{1}{p_0+2}} \quad \text{or} \quad 0 < \epsilon_k << \Big(a^* - \frac{a_{1k} + a_{2k}}{2} \Big)^{\frac{1}{p_0+2}} \quad \text{as} \quad k \to \infty,$$

it then follows from (5.35) that $e(a_{1k}, a_{2k}) >> \Big(a^* - \frac{a_{1k}+a_{2k}}{2} \Big)^{\frac{p_0}{p_0+2}}$ as $k \to \infty$, which however contradicts (5.3). This completes the proof of (5.4).

Since the above argument shows that Proposition 5.1 holds for any given subsequence $\{(a_{1k}, a_{2k})\}$ with $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$, an approach similar to that of [16] then gives that Proposition 5.1 holds essentially for the whole sequence $\{(a_1, a_2)\}$ satisfying $(a_1, a_2) \nearrow (a^*, a^*)$. $\square$

6 Proof of Theorem 1.5

This section is devoted to the proof of Theorem 1.5 on the mass concentration of minimizers. Using the notations as in (5.1)-(5.2), in order to prove Theorem 1.5 on the minimizers of (1.2) as $(b_1+\beta, b_2+\beta) \nearrow (a^*, a^*)$, it suffices to establish the following theorem on the minimizers of (5.2) as $(a_1, a_2) \nearrow (a^*, a^*)$.

Theorem 6.1. *Assume that $0<\beta<a^*$ and $V_i(x)$ satisfies (1.17) and (1.18) for $i=1$ and 2. For any sequence $\{(a_{1k}, a_{2k})\}$ satisfying $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$ as $k\to\infty$, and let (u_{a_1k}, u_{a_2k}) be the corresponding non-negative minimizer of (5.2). Then there exists a subsequence of $\{(a_{1k}, a_{2k})\}$, still denoted by $\{(a_{1k}, a_{2k})\}$, such that each $u_{a_{ik}}$ ($i=1$, 2) has a unique global maximum point x_{ik} satisfying*

$$x_{ik} \longrightarrow \bar{x}_0 \in \mathcal{Z} \text{ and } \frac{|x_{ik}-\bar{x}_0|}{\big(a^*-\frac{a_{1k}+a_{2k}}{2}\big)^{\frac{1}{p_0+2}}} \longrightarrow 0 \quad \text{as } k\to\infty. \tag{6.1}$$

Moreover, for $i=1$ and 2,

$$\lim_{k\to\infty}\Big(a^*-\frac{a_{1k}+a_{2k}}{2}\Big)^{\frac{1}{p_0+2}} u_{a_{ik}}\Big(\big(a^*-\frac{a_{1k}+a_{2k}}{2}\big)^{\frac{1}{p_0+2}}x+x_{ik}\Big) = \frac{\lambda}{\|Q\|_2}Q(\lambda x)$$

strongly in $H^1(\mathbb{R}^2)$, where $\lambda>0$ is given by

$$\lambda = \Big(\frac{p_0\gamma}{4}\int_{\mathbb{R}^2}|x|^{p_0}Q^2(x)dx\Big)^{\frac{1}{p_0+2}}, \tag{6.2}$$

$p_0>0$ and $\gamma>0$ are defined in (1.20)-(1.23).

Let (u_{a_1}, u_{a_2}) be a non-negative minimizer of (5.2), where $(a_1, a_2)\nearrow(a^*, a^*)$. Define

$$\varepsilon := \big(a^*-\frac{a_1+a_2}{2}\big)^{\frac{1}{p_0+2}} > 0. \tag{6.3}$$

It then follows from (5.4) and (5.5) that, as $(a_1, a_2)\nearrow(a^*, a^*)$,

$$\sum_{i=1}^{2}\int_{\mathbb{R}^2}V_i(x)|u_{a_i}(x)|^2dx \le e(a_1, a_2) < C\big(a^*-\frac{a_1+a_2}{2}\big)^{\frac{p_0}{p_0+2}} \tag{6.4}$$

and

$$\int_{\mathbb{R}^2}|\nabla u_{a_i}(x)|^2dx \sim \varepsilon^{-2}, \quad \int_{\mathbb{R}^2}|u_{a_i}(x)|^4dx \sim \varepsilon^{-2}, \tag{6.5}$$

where $i=1$, 2. Similar to (5.21), we know that there exist a sequence $\{y_\varepsilon\}$ as well as positive constants R_0 and η_i such that

$$\liminf_{\varepsilon\searrow 0}\int_{B_{R_0}(0)}|w_{a_i}|^2dx \ge \eta_i > 0, \quad i=1,\ 2, \tag{6.6}$$

where w_{a_i} is the $L^2(\mathbb{R}^2)$-normalized function defined by

$$w_{a_i}(x) = \varepsilon u_{a_i}(\varepsilon x+\varepsilon y_\varepsilon), \quad i=1,\ 2. \tag{6.7}$$

Note from (6.5) that

$$M \le \int_{\mathbb{R}^2} |\nabla w_{a_i}|^2 dx \le \frac{1}{M}, \quad M \le \int_{\mathbb{R}^2} |w_{a_i}|^4 dx \le \frac{1}{M}, \quad i = 1,\, 2, \tag{6.8}$$

where the positive constant M is independent of a_1 and a_2.

Proof of Theorem 6.1. Let $\varepsilon_k := \big(a^* - \frac{a_{1k}+a_{2k}}{2}\big)^{\frac{1}{p_0+2}} > 0$, where $(a_{1k}, a_{2k}) \nearrow (a^*, a^*)$ as $k \to \infty$, and denote $(u_{1k}(x), u_{2k}(x)) := (u_{a_{1k}}(x), u_{a_{2k}}(x))$ a non-negative minimizer of (5.2). Inspired by [14, 37], we shall complete the proof of Theorem 6.1 by the following three steps.

Step 1: Decay estimate for $(u_{1k}(x), u_{2k}(x))$. Let $w_{ik}(x) := w_{a_{ik}}(x) \ge 0$ be defined by (6.7). By a similar analysis to the proof of Lemma 5.3(ii), we know that there exists a subsequence of $\{\varepsilon_k\}$, still denoted by $\{\varepsilon_k\}$, such that

$$z_k := \varepsilon_k y_{\varepsilon_k} \xrightarrow{k} x_0 \ \text{ for some } \ x_0 \in \Lambda, \tag{6.9}$$

where the set Λ is defined by (1.19). and w_{ik} $(i = 1, 2)$ satisfies

$$\begin{cases} -\Delta w_{1k} + \varepsilon_k^2 V_1(\varepsilon_k x + z_k) w_{1k} = \varepsilon_k^2 \mu_{1k} w_{1k} + a_{1k} w_{1k}^3 - \beta(w_{1k}^2 - w_{2k}^2) w_{1k} \ \text{ in } \ \mathbb{R}^2, \\ -\Delta w_{2k} + \varepsilon_k^2 V_2(\varepsilon_k x + z_k) w_{2k} = \varepsilon_k^2 \mu_{2k} w_{2k} + a_{2k} w_{2k}^3 - \beta(w_{2k}^2 - w_{1k}^2) w_{2k} \ \text{ in } \ \mathbb{R}^2, \end{cases} \tag{6.10}$$

where (μ_{1k}, μ_{2k}) is a suitable Lagrange multiplier satisfying

$$\mu_{ik} \sim -\varepsilon_k^{-2} \ \text{ and } \ \mu_{1k}/\mu_{2k} \to 1 \quad \text{as} \quad (a_{1k}, a_{2k}) \nearrow (a^*, a^*), \quad i = 1,\, 2.$$

Moreover, $w_{ik} \xrightarrow{k} w_0$ strongly in $H^1(\mathbb{R}^2)$ for some $w_0 > 0$ satisfying

$$-\Delta w_0(x) = -\lambda^2 w_0(x) + a^* w_0^3(x) \quad \text{ in } \ \mathbb{R}^2, \tag{6.11}$$

where $\lambda > 0$ is a constant. Similar to (5.33), we know that

$$w_0(x) = \frac{\lambda}{\|Q\|_2} Q(\lambda|x - y_0|) \ \text{ for some } \ y_0 \in \mathbb{R}^2. \tag{6.12}$$

By the exponential decay of Q, then for any $\alpha > 2$,

$$\int_{|x|\ge R} |w_{ik}|^\alpha dx \to 0 \ \text{ as } \ R \to \infty \ \text{ uniformly for large } k, \ \text{ where } \ i = 1,\, 2. \tag{6.13}$$

Recall from (6.10) that $-\Delta w_{ik}(x) \le c_i(x) w_{ik}(x)$ in $\mathbb{R}^2$, where $c_i(x) = a_{ik} w_{ik}^2 + (-1)^i \beta(w_{1k}^2 - w_{2k}^2)$ in $\mathbb{R}^2$ for $i = 1,\, 2$. By applying De Giorgi-Nash-Moser theory, we thus have

$$\max_{B_1(\xi)} w_{ik} \le C\Big(\int_{B_2(\xi)} |w_{ik}|^\alpha dx\Big)^{\frac{1}{\alpha}}, \ \ i = 1,\, 2,$$

where ξ is an arbitrary point in $\mathbb{R}^2$, and $C > 0$ is a constant depending only on the bound of $\|w_{1k}\|_{L^\alpha(B_2(\xi))} + \|w_{2k}\|_{L^\alpha(B_2(\xi))}$. Hence (6.13) implies that

$$w_{ik}(x) \to 0 \quad \text{as} \quad |x| \to \infty \quad \text{uniformly for } k, \quad i = 1,\, 2. \tag{6.14}$$

Since w_{ik} $(i = 1, 2)$ satisfies (6.10), apply the comparison principle as in [21] to compare w_{ik} with $Ce^{-\frac{\lambda}{2}|x|}$ ($\lambda > 0$ is obtained in (6.11)), which then shows that there exist a constant $C > 0$ and a large constant $R > 0$, independent of k, such that

$$w_{ik}(x) \le Ce^{-\frac{\lambda}{2}|x|} \quad \text{for} \quad |x| > R \quad \text{as} \quad k \to \infty, \quad i = 1, 2. \tag{6.15}$$

Step 2: The detailed concentration behavior. For the convergent subsequence $\{w_{ik}(x)\}$ obtained in Step 1, let $\bar z_{ik}$ be any local maximum point of $u_{ik}(x)$, $(i = 1, 2)$. We claim that there is a sequence $\{k\}$, passing to a subsequence if necessary, such that

$$\lim_{k\to\infty} \frac{|\bar z_{1k} - \bar z_{2k}|}{\varepsilon_k} = 0\,. \tag{6.16}$$

For showing (6.16), we first prove that

$$\limsup_{k\to\infty} \frac{|\bar z_{ik} - z_k|}{\varepsilon_k} < \infty, \quad i = 1, 2, \tag{6.17}$$

where $z_k = \varepsilon_k y_{\varepsilon_k} x_0 \in \Lambda$ as $k \to \infty$. Indeed, if (6.17) is false, i.e., $\frac{|\bar z_{ik} - z_k|}{\varepsilon_k} \stackrel{k}{\to} \infty$ holds for $i = 1$ or 2. Without loss of generality, we assume that $\frac{|\bar z_{1k} - z_k|}{\varepsilon_k} \stackrel{k}{\to} \infty$. It follows from (6.7) and (6.15) that

$$u_{ik}(\bar z_{1k}) = \frac{1}{\varepsilon_k} w_{ik}\big(\frac{\bar z_{1k} - z_k}{\varepsilon_k}\big) = o\big(\frac{1}{\varepsilon_k}\big) \quad \text{as} \quad k \to \infty, \quad i = 1, 2.$$

This however leads to a contradiction, since (5.27) implies that

$$a_{1k} u_{1k}^2(\bar z_{1k}) - \beta\big(u_{1k}^2(\bar z_{1k}) - u_{2k}^2(\bar z_{1k})\big) \ge -\mu_{1k} \ge C\varepsilon_k^{-2}.$$

Therefore, (6.17) is true.

By (6.17), there exists a sequence $\{k\}$ such that

$$\lim_{k\to\infty} \frac{\bar z_{ik} - z_k}{\varepsilon_k} = y_i \quad \text{for some } \; y_i \in \mathbb{R}^2\,, \;\; i = 1, 2.$$

Set

$$\bar w_{ik}(x) := w_{ik}\big(x + \frac{\bar z_{ik} - z_k}{\varepsilon_k}\big) = \varepsilon_k u_{ik}\big(\varepsilon_k x + \bar z_{ik}\big), \quad i = 1, 2. \tag{6.18}$$

By Step 1, $w_{ik} \to w_0$ strongly in $H^1(\mathbb{R}^2)$ as $k \to \infty$, and $w_0 > 0$ satisfying (6.12), then

$$\lim_{k\to\infty} \bar w_{ik}(x) = w_0(x + y_i) = \frac{\lambda}{\|Q\|_2} Q(\lambda|x + y_i - y_0|) \text{ strongly in } H^1(\mathbb{R}^2), \;\; i = 1, 2.$$

Since the origin $(0, 0)$ is a critical point of $\bar w_{ik}$ for all $k > 0$ $(i = 1, 2)$, it is also a critical point of $w_0(x + y_i)$. On the other hand, $Q(\lambda|x - z_0|)$ possesses z_0 as its unique critical (maximum) point. We therefore conclude that $w_0(x + y_i) = \frac{\lambda}{\|Q\|_2} Q(\lambda|x + y_i - y_0|)$ is spherically symmetric about the origin. Hence, $y_i = y_0$, and

$$\lim_{k\to\infty} \bar w_{ik}(x) = \frac{\lambda}{\|Q\|_2} Q(\lambda|x|) := \bar w_0 \text{ strongly in } H^1(\mathbb{R}^2), \;\; i = 1, 2. \tag{6.19}$$

The estimate (6.16) is followed by (6.18) and (6.19).

Similar to the discussion of proving (6.17), we know that each local maximum point of $\bar{w}_{ik}(x)$ ($i = 1, 2$), which is also a local maximum point of $u_{ik}(x)$, must stay in a finite ball in $\mathbb{R}^2$. Since $V_i(x) \in C^{\alpha}_{\rm loc}(\mathbb{R}^2)$, we can deduce from (6.10) and standard elliptic regular theory that

$$w_{ik}(x) \xrightarrow{k} w_0(x) \ \text{ in } \ C^{2,\alpha}_{loc}(\mathbb{R}^2), \ \ i = 1, 2.$$

Moreover, by (6.18) and (6.16), we can further obtain that

$$\bar{w}_{ik}(x) \xrightarrow{k} \bar{w}_0(x) \ \text{ in } \ C^{2,\alpha}_{loc}(\mathbb{R}^2), \ \ i = 1, 2.$$

Because the origin $(0, 0)$ is the only non-degenerate critical point of $\bar{w}_0(x)$, all local maximum points of $\bar{w}_{ik}(x)$ must approach the origin and stay in a small ball $B_\epsilon(0)$ as $k \to \infty$, where $\epsilon > 0$ is small. It then follows from Lemma 4.2 in [30] that for large k, $\bar{w}_{ik}(x)$ has no critical points other than the origin. This gives the uniqueness of local maximum points for $\bar{w}_{ik}(x)$ and $u_{ik}(x)$ ($i = 1, 2$).

Step 3: Completion of the proof. Let

$$\gamma_j(x) = \frac{V_1(x) + V_2(x)}{|x - x_{1j}|^{\bar{p}_j}}, \quad 1 \le j \le l, \tag{6.20}$$

where $\bar{p}_j > 0$ is defined by (1.20), so that the limit $\lim_{x \to x_{1j}} \gamma_j(x) = \gamma_j(x_{1j})$ exists for all $i \in \{1, \cdots, l\}$ in view of the assumptions on V_1 and V_2. Moreover, one can note that $\gamma_j(x_{1j}) = \gamma_j \ge \gamma$ if $x_{1j} \in \bar{\Lambda}$, where γ_j and $\bar{\Lambda}$ are defined by (1.22) and (1.21), respectively. We hence obtain from (6.18) that

$$\begin{aligned}
e(a_{1k}, a_{2k}) &= E_{a_{1k},a_{2k}}(u_{a_{1k}}, u_{a_{2k}}) \\
&\ge \sum_{i=1}^{2} \Big\{ \frac{1}{\varepsilon_k^2} \Big[\int_{\mathbb{R}^2} |\nabla \bar{w}_{ik}(x)|^2 dx - \frac{a^*}{2} \int_{\mathbb{R}^2} |\bar{w}_{ik}(x)|^4 dx \Big] \\
&\quad + \frac{a^* - a_{ik}}{2\varepsilon_k^2} \int_{\mathbb{R}^2} |\bar{w}_{ik}(x)|^4 dx + \int_{\mathbb{R}^2} V_i(\varepsilon_k x + \bar{z}_{ik}) |\bar{w}_{ik}(x)|^2 dx \Big\} \\
&\ge \sum_{i=1}^{2} \Big[\frac{a^* - a_{ik}}{2\varepsilon_k^2} \int_{\mathbb{R}^2} |\bar{w}_{ik}(x)|^4 dx + \int_{\mathbb{R}^2} V_i(\varepsilon_k x + \bar{z}_{ik}) |\bar{w}_{ik}(x)|^2 dx \Big],
\end{aligned} \tag{6.21}$$

where $\bar{z}_{ik}$ is the unique global maximum point of u_{ik}, and $\bar{z}_{ik} \xrightarrow{k} x_0 \in \Lambda$, $i = 1, 2$. We may assume that $x_0 = x_{1j_0}$ for some $1 \le j_0 \le l$.

We first claim that

$$\frac{|\bar{z}_{ik} - x_{1j_0}|}{\varepsilon_k} \ \text{ is uniformly bounded as } \ k \to \infty, \quad \text{where } \ i = 1, 2. \tag{6.22}$$

Otherwise, if (6.22) is false for $i = 1$ or 2. It then follows from (6.16) that both of them are unbounded, and hence there exists a subsequence of $\{(a_{1k}, a_{2k})\}$, still denoted by $\{(a_{1k}, a_{2k})\}$, such that

$$\lim_{k \to \infty} \frac{|\bar{z}_{ik} - x_{1j_0}|}{\varepsilon_k} = \infty, \quad i = 1, 2.$$

We then derive from Fatou's Lemma that for any $M > 0$ large enough,

$$
\begin{aligned}
&\liminf_{k\to\infty} \varepsilon_k^{-\bar{p}_{j_0}} \sum_{i=1}^{2} \int_{\mathbb{R}^2} V_i(\varepsilon_k x + \bar{z}_{ik}) |\bar{w}_{ik}(x)|^2 dx \\
=& \liminf_{k\to\infty} \sum_{i=1}^{2} \int_{\mathbb{R}^2} \frac{V_i(\varepsilon_k x + \bar{z}_{ik})}{|\varepsilon_k x + \bar{z}_{ik} - x_{1j_0}|^{\bar{p}_{j_0}}} \Big| x + \frac{\bar{z}_{ik} - x_{1j_0}}{\varepsilon_k} \Big|^{\bar{p}_{j_0}} |\bar{w}_{ik}(x)|^2 dx \\
\geq& \sum_{i=1}^{2} \int_{\mathbb{R}^2} \liminf_{k\to\infty} \Big(\frac{V_i(\varepsilon_k x + \bar{z}_{ik})}{|\varepsilon_k x + \bar{z}_{ik} - x_{1j_0}|^{\bar{p}_{j_0}}} \Big| x + \frac{\bar{z}_{ik} - x_{1j_0}}{\varepsilon_k} \Big|^{\bar{p}_{j_0}} |\bar{w}_{ik}(x)|^2 \Big) dx \geq M.
\end{aligned}
$$

This estimate and (6.21) imply that

$$
e(a_{1k}, a_{2k}) \geq M \varepsilon_k^{\bar{p}_{j_0}} = M \Big(a^* - \frac{a_{1k} + a_{2k}}{2} \Big)^{\bar{p}_{j_0}} \tag{6.23}
$$

holds for arbitrary constant $M > 0$, which however contradicts Proposition 5.1, due to the fact that $\bar{p}_{j_0} \leq p_0$. Therefore, (6.22) is proved.

We next show that $\bar{p}_{j_0} = p_0$, i.e., $x_{1j_0} \in \bar{\Lambda}$, where the set $\bar{\Lambda}$ is defined by (1.21). By (6.22), we know that there exists a subsequence of $\{(a_{1k}, a_{2k})\}$ such that

$$
\lim_{k\to\infty} \frac{\bar{z}_{ik} - x_{1j_0}}{\varepsilon_k} = \bar{z}_0 \quad \text{for some } \ \bar{z}_0 \in \mathbb{R}^2, \quad i = 1,\, 2. \tag{6.24}
$$

Since Q is a radially decreasing function and decays exponentially as $|x| \to \infty$, we then deduce from (6.19) that

$$
\begin{aligned}
&\liminf_{k\to\infty} \varepsilon_k^{-\bar{p}_{j_0}} \sum_{i=1}^{2} \int_{\mathbb{R}^2} V_i(\epsilon_k x + \bar{z}_{ik}) |\bar{w}_{ik}(x)|^2 dx \\
=& \liminf_{k\to\infty} \sum_{i=1}^{2} \int_{\mathbb{R}^2} \frac{V_i(\epsilon_k x + \bar{z}_{ik})}{|\varepsilon_k x + \bar{z}_{ik} - x_{1j_0}|^{\bar{p}_{j_0}}} \Big| x + \frac{\bar{z}_{ik} - x_{1j_0}}{\varepsilon_k} \Big|^{\bar{p}_{j_0}} |\bar{w}_{ik}(x)|^2 dx \\
\geq& \gamma_{j_0}(x_{1j_0}) \int_{\mathbb{R}^2} |x + \bar{z}_0|^{\bar{p}_{j_0}} \bar{w}_0^2 dx \geq \frac{\gamma_{j_0}(x_{1j_0})}{\lambda^{\bar{p}_{j_0}} \|Q\|_2^2} \int_{\mathbb{R}^2} |x|^{\bar{p}_{j_0}} Q^2 dx,
\end{aligned} \tag{6.25}
$$

where the equality holds if and only if $\bar{z}_0 = (0, 0)$. We hence obtain from (6.21) and (6.25) that $\bar{p}_{j_0} = p_0$, otherwise we get that (6.23) holds with M being replaced by some $C > 0$, which however contradicts Proposition 5.1.

By the fact $\bar{p}_{j_0} = p_0$, we now have $x_{1j_0} \in \bar{\Lambda}$ and $\gamma_{j_0}(x_{1j_0}) = \gamma_{j_0}$. It then follows from (6.3), (6.21) and (6.25) as well as (1.11)that

$$
\begin{aligned}
\liminf_{k\to\infty} \frac{e(a_{1k}, a_{2k})}{\varepsilon_k^{p_0}} &\geq \|\bar{w}_0\|_4^4 + \gamma_{j_0} \int_{\mathbb{R}^2} |x + \bar{z}_0|^{p_0} \bar{w}_0^2 dx \\
&\geq \frac{1}{a^*} \Big(2\lambda^2 + \frac{\gamma_{j_0}}{\lambda^{p_0}} \int_{\mathbb{R}^2} |x|^{p_0} Q^2 dx \Big),
\end{aligned} \tag{6.26}
$$

and"=" holds in the last inequality if and only if $\bar{z}_0 = (0, 0)$. The estimate (6.1) then follows from (6.24) and this conclusion. Further, taking the infimum of (6.26) over $\lambda > 0$ yields that

$$
\liminf_{k\to\infty} \frac{e(a_{1k}, a_{2k})}{\varepsilon_k^{p_0}} \geq \frac{2p_0 + 4}{p_0 a^*} \Big(\frac{p_0 \gamma_{j_0} \int_{\mathbb{R}^2} |x|^{p_0} Q^2 dx}{4} \Big)^{\frac{2}{p_0+2}}, \tag{6.27}
$$

where the equality is achieved at

$$\lambda = \lambda_0 := \Big(\frac{p_0\gamma_{j_0}\int_{\mathbb{R}^2}|x|^{p_0}Q^2dx}{4}\Big)^{\frac{1}{p_0+2}}.$$

We finally remark that the limit of (6.27) actually exists and is equal to the right hand side of (6.27). To see this fact, we simply take

$$u_1(x) = u_2(x) = \frac{\alpha}{\varepsilon_k\|Q\|_2}Q\Big(\frac{\alpha|x-x_{1j_1}|}{\varepsilon_k}\Big) \quad \text{with} \quad x_{1j_1}\in\mathcal{Z}$$

as a trial function for $E_{a_{1k},a_{2k}}(\cdot,\cdot)$ and minimizes it over $\alpha>0$, which then leads to the limit

$$\liminf_{k\to\infty}\frac{e(a_{1k},a_{2k})}{\varepsilon_k^{p_0}} \leq \frac{2p_0+4}{p_0a^*}\Big(\frac{p_0\gamma\int_{\mathbb{R}^2}|x|^{p_0}Q^2dx}{4}\Big)^{\frac{2}{p_0+2}}. \tag{6.28}$$

Since $\gamma=\min\{\gamma_1,\dots,\gamma_l\}$, it follows from (6.27) and (6.28) that $\gamma=\gamma_{j_0}$, i.e., $x_{1j_0}\in\mathcal{Z}$, and further (6.27) is indeed an equality. This yields that λ is unique, which is independent of the choice of the subsequence, and minimizes (6.26), i.e., $\lambda=\lambda_0$. Moreover, when (6.27) becomes an equality, which implies that (6.26) is also an equality. Thus, $\bar{z}_0=(0,0)$, which together with (6.24) give (6.1). $\square$

A Appendix: Some Proofs

In this appendix we shall establish the following lemma and provide a different proof of Theorem 1.1.

Lemma A.1. *Suppose that positive constants b_1, b_2 and β satisfy $0<b_1<b_2<a^*$ and $0<b_2\leq 2\beta+b_1$. Define*

$$l(t) = \frac{t^2+t}{\frac{b_2}{2}t^2+\beta t+\frac{b_1}{2}}, \quad t\in[0,\infty).$$

Then we have

$$l(t) > l(1) = \frac{2}{\frac{b_2}{2}+\beta+\frac{b_1}{2}}, \quad t\in(1,\infty).$$

Proof. Direct calculation shows that

$$l'(t) = \frac{(\beta-\frac{b_2}{2})t^2+b_1t+\frac{b_1}{2}}{\big(\frac{b_2}{2}t^2+\beta t+\frac{b_1}{2}\big)^2}. \tag{A.1}$$

Let

$$m(t) = (\beta-\frac{b_2}{2})t^2+b_1t+\frac{b_1}{2}, \ t\in[0,\infty).$$

If $\beta\geq\frac{b_2}{2}$, then $m'(t)=2(\beta-\frac{b_2}{2})t+b_1>0$. This implies that $l'(t)>0$ for $t\in[0,\infty)$, and we are done.

We now suppose that $\beta < \frac{b_2}{2}$. In this case, we have

$$m(t) > 0 \quad \text{if} \quad t \in (0, \hat{t}\,); \;\; m(t) < 0 \quad \text{if} \quad t \in (\hat{t}, \infty),$$

where

$$\hat{t} = \frac{b_1 + \sqrt{b_1(b_1 + b_2 - 2\beta)}}{b_2 - 2\beta} > 1,$$

in view of the assumption that $b_2 \leq 2\beta + b_1$. Thus, $l(t)$ is strictly increasing in $(1, \hat{t}\,]$ and strictly decreasing in $(\hat{t}, \infty)$. Since $b_2 \leq 2\beta + b_1$, we thus conclude that for any $t_0 > 1$,

$$l(t_0) > \lim_{t\to\infty} l(t) = \frac{2}{b_2} \geq l(1) = \frac{2}{\frac{b_2}{2} + \beta + \frac{b_1}{2}},$$

and the proof is therefore complete. □

Inspired by [14], in the following we reprove Theorem 1.1 by using the Gagliardo-Nirenberg inequality (1.10) and some recaling techniques. For the reader's convenience, we restate Theorem 1.1 as the following lemma.

Lemma A.2. *Let Q be the unique positive radial solution of (1.9) and suppose $V_i(x)$ satisfies (1.6) for $i = 1, 2$. Then,*

(i) If $0 < b_1 < a^*$, $0 < b_2 < a^*$ and $0 < \beta < \sqrt{(a^* - b_1)(a^* - b_2)}$, then there exists at least one minimizer for (1.2).

(ii) If either $b_1 > a^*$ or $b_2 > a^*$ or $\beta > \frac{a^*-b_1}{2} + \frac{a^*-b_2}{2}$, then there is no minimizer for (1.2).

Proof. (i) : We first note from the Gagliardo-Nirenberg inequality (1.10) that for any $(u_1, u_2) \in \mathcal{X}$ satisfying $\|u_1\|_2^2 = \|u_2\|_2^2 = 1$,

$$E_{b_1,b_2,\beta}(u_1, u_2) \geq \sum_{i=1}^{2} \int_{\mathbb{R}^2} \Big[\Big(\frac{a^* - b_i}{2}\Big)|u_i|^4 + V_i(x)|u_i|^2\Big]dx - \beta \int_{\mathbb{R}^2} |u_1|^2|u_2|^2 dx. \quad \text{(A.2)}$$

Since $V_i(x) \geq 0$ and $0 < \beta < \sqrt{(a^* - b_1)(a^* - b_2)}$, one can deduce that $\hat{e}(b_1, b_2, \beta) \geq 0$ for all $0 < b_i < a^* := \|Q\|_2^2$ $(i = 1, 2)$. Let $\{(u_{1n}, u_{2n})\} \subset \mathcal{M}$ be a minimizing sequence of problem (1.2) satisfying

$$\|u_{1n}\|_2^2 = \|u_{2n}\|_2^2 = 1 \quad \text{and} \quad \lim_{n\to\infty} E_{b_1,b_2,\beta}(u_{1n}, u_{2n}) = \hat{e}(b_1, b_2, \beta).$$

Taking $\delta \in (\frac{\beta}{a^*-b_2}, \frac{a^*-b_1}{\beta})$, it then follows from (A.2) and Young's inequality that

$$\begin{aligned}\hat{e}(b_1, b_2, \beta) + 1 &\geq \sum_{i=1}^{2} \int_{\mathbb{R}^2} \Big(\frac{a^* - b_i}{2}|u_{in}|^4 + V_i(x)|u_{in}|^2\Big)dx - \beta \int_{\mathbb{R}^2} |u_{1n}|^2|u_{2n}|^2 dx \\ &\geq \Big(\frac{a^* - b_1}{2} - \frac{\beta\delta}{2}\Big) \int_{\mathbb{R}^2} |u_{1n}|^4 dx + \Big(\frac{a^* - b_2}{2} - \frac{\beta}{2\delta}\Big) \int_{\mathbb{R}^2} |u_{2n}|^4 dx.\end{aligned}$$

This implies that $\{(u_{1n}, u_{2n})\}$ is bounded in $L^4(\mathbb{R}^2) \times L^4(\mathbb{R}^2)$ uniformly w.r.t. n, and it is then easy to deduce that $\{(u_{1n}, u_{2n})\}$ is bounded uniformly in $\mathcal{X}$. Therefore, by the

compactness of Lemma 2.1, there exist a subsequence of $\{(u_{1n}, u_{2n})\}$ and $(u_1, u_2) \in \mathcal{X}$ such that

$$\begin{aligned}
(u_{1n}, u_{2n}) &\overset{n}{\rightharpoonup} (u_1, u_2) \quad \text{weakly in } \mathcal{X}\,,\\
(u_{1n}, u_{2n}) &\overset{n}{\to} (u_1, u_2) \quad \text{strongly in } L^q(\mathbb{R}^2) \times L^q(\mathbb{R}^2),
\end{aligned}$$

where $2 \le q < \infty$. We therefore have $\|u_1\|_2^2 = \|u_2\|_2^2 = 1$ and $E_{b_1,b_2,\beta}(u_1, u_2) = \hat{e}(b_1, b_2, \beta)$. This proves the existence of minimizers for the case where $0 < b_1 < a^*$, $0 < b_2 < a^*$ and $0 < \beta < \sqrt{(a^* - b_1)(a^* - b_2)}$.

(ii) : Let $\varphi(x) \in C_0^\infty(\mathbb{R}^2)$ be a nonnegative smooth cutoff function such that $\varphi(x) = 1$ if $|x| \le 1$ and $\varphi(x) = 0$ if $|x| \ge 2$. For any $\bar{x}_0 \in \mathbb{R}^2$, $\tau > 0$ and $R > 0$, set

$$\phi(x) = A_{R\tau} \frac{\tau}{\|Q\|_2} \varphi\Big(\frac{x - \bar{x}_0}{R}\Big) Q\big(\tau|x - \bar{x}_0|\big), \tag{A.3}$$

where $A_{R\tau} > 0$ is chosen such that $\int_{\mathbb{R}^2} \phi^2 dx = 1$. By scaling, $A_{R\tau}$ depends only on the product $R\tau$. In fact, using the exponential decay of Q in (1.12), we have

$$\frac{1}{A_{R\tau}^2} = \frac{1}{\|Q\|_2^2} \int_{\mathbb{R}^2} \varphi^2\Big(\frac{x}{R\tau}\Big) Q^2(x) dx = 1 + O((R\tau)^{-\infty}) \quad \text{as} \quad R\tau \to \infty. \tag{A.4}$$

Here we use the notation $f(t) = O(t^{-\infty})$ for a function f satisfying $\lim_{t\to\infty} |f(t)| t^s = 0$ for all $s > 0$. By the exponential decay of $Q(x)$ and the equality (1.11), we have

$$\begin{aligned}
\int_{\mathbb{R}^2} |\nabla\phi|^2 - \frac{b_i}{2} \int_{\mathbb{R}^2} \phi^4 dx &= \frac{\tau^2}{\|Q\|_2^2} \int_{\mathbb{R}^2} |\nabla Q|^2 - \frac{b_i \tau^2}{2\|Q\|_2^4} \int_{\mathbb{R}^2} Q^4 dx + O((R\tau)^{-\infty})\\
&= \Big(1 - \frac{b_i}{\|Q\|_2^2}\Big)\tau^2 + O((R\tau)^{-\infty}) \quad \text{as} \quad R\tau \to \infty.
\end{aligned} \tag{A.5}$$

On the other hand, since the function $x \mapsto V_i(x)\varphi^2(\frac{x-\bar{x}_0}{R})$ is bounded and has compact support, we obtain that

$$\lim_{\tau\to\infty} \int_{\mathbb{R}^2} V_i(x)\phi^2(x) dx = V_i(\bar{x}_0) \tag{A.6}$$

holds for almost every $\bar{x}_0 \in \mathbb{R}^2$, where $i = 1, 2$.

Suppose that $b_1 > a^*$. Choosing $\eta(x) \in C_0^\infty(\mathbb{R}^2)$ such that $\int_{\mathbb{R}^2} \eta^2(x) dx = 1$, we then derive that

$$\int_{\mathbb{R}^2} \phi^2 \eta^2 dx \le \sup_{x \in R^2} \eta^2(x) \int_{\mathbb{R}^2} \phi^2 dx = \sup_{x \in R^2} \eta^2(x) < \infty,$$

where $\phi > 0$ is as in (A.3). Together with (A.5) and (A.6), this estimate then yields that

$$E_{b_1,b_2,\beta}(\phi, \eta) \le \int_{\mathbb{R}^2} |\nabla\phi|^2 - \frac{b_1}{2} \int_{\mathbb{R}^2} \phi^4 dx + C \le \frac{a^* - b_1}{a^*}\tau^2 + C,$$

which implies that

$$\hat{e}(b_1, b_2, \beta) \le \lim_{\tau\to\infty} E_{b_1,b_2,\beta}(\phi, \eta) = -\infty.$$

Similarly, this estimate is still true if $b_2 > a^*$. Therefore, $\hat{e}(b_1, b_2, \beta)$ does not admit any minimizer.

Finally, if $\beta > \frac{a^*-b_1}{2} + \frac{a^*-b_2}{2}$, we also deduce from (A.5) and (A.6) that

$$\hat{e}(b_1, b_2, \beta) \leq \lim_{\tau \to \infty} E_{b_1,b_2,\beta}(\phi, \phi) = -\infty,$$

which also implies the non-existence of minimizers. This completes the proof of Lemma A.2. $\square$

Acknowledgements: This work was supported by National Natural Science Foundation of China (11322104, 11271360, 11471331, 11501555) and National Center for Mathematics and Interdisciplinary Sciences.

Conflict of interest statement: The authors declare that they have no conflict of interest.

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